Exercise Problem Sets 1

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Problem 1. Determine whether the sequence $\{a_n\}_{n=1}^{\infty}$ converges or diverges. If it converges, find the limit.

(1) $a_n = \dfrac{\ln n}{\ln(2n)}$ (2) $a_n = \dfrac{(-1)^{n+1}n}{n+\sqrt{n}}$ (3) $a_n = n\sin\dfrac{1}{n}$ (4) $a_n = n - \sqrt{n+1}\sqrt{n+3}$

(5) $a_n = \sqrt[n]{n^2+n}$ (6) $a_n = (3^n+5^n)^{\frac{1}{n}}$ (7) $a_n = \dfrac{1}{\sqrt{n^2-1}-\sqrt{n^2+n}}$

(8) $a_n = \sqrt{n}\ln\big(1+\dfrac{1}{n}\big)$ (9) $a_n = \dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2^n n!}$ (10) $a_n = \dfrac{1\cdot 3\cdot 5\cdot\cdots\cdot(2n-1)}{2^n(n+1)!}$.

Problem 2. Determine whether the series $\sum_{n=1}^{\infty} a_n$ is convergent or divergent. If it is convergent, find its sum.

(1) $a_n = \dfrac{1}{1+(\frac{2}{3})^n}$ (2) $a_n = \ln\big(\dfrac{n^2+1}{2n^2+1}\big)$ (3) $a_n = e^{-n} + \dfrac{1}{n(n+1)}$ (4) $a_n = \dfrac{1}{n^3-n}$

(5) $a_n = \dfrac{40n}{(2n-1)^2(2n+1)^2}$

Problem 3. Find values of x for which the following series converges.

(1) $\sum_{n=1}^{\infty}(-4)^n(x-5)^n$ (2) $\sum_{n=1}^{\infty}\dfrac{2^n}{x^n}$ (3) $\sum_{n=1}^{\infty}\dfrac{\sin^n x}{3^n}$ (4) $\sum_{n=1}^{\infty}e^{nx}$.

Problem 4. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers.

(1) Show that if $\lim_{n\to\infty}(a_n+b_n)$ D.N.E. and $\lim_{n\to\infty} b_n$ converges, then $\lim_{n\to\infty} a_n$ D.N.E.

(2) Show that if $\sum_{n=1}^{\infty}(a_n+b_n)$ diverges and $\sum_{n=1}^{\infty} b_n$ converges, then $\sum_{n=1}^{\infty} a_n$ diverges.

Problem 5. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers, and $\{\sigma_n\}_{n=1}^{\infty}$ be a sequence of real numbers defined by

$$\sigma_n = \frac{a_1+a_2+\cdots+a_n}{n} = \frac{1}{n}\sum_{k=1}^{n} a_k\,.$$

(1) Show that if $\lim_{n\to\infty} a_n = a$ exists, then $\lim_{n\to\infty}\sigma_n = a$.

(2) Suppose that $\lim_{n\to\infty}\sigma_n = a$ exists, is it necessary that $\lim_{n\to\infty} a_n = a$?

Problem 6. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real numbers defined recursively by

$$a_{n+1} = \sqrt{1+a_n} \quad \forall\, n\in\mathbb{N}\cup\{0\}\,, a_0 = 0\,.$$

Show that $\{a_n\}_{n=1}^{\infty}$ converges and find the limit.

Problem 7. Let $a_n = \big(1+\dfrac{1}{n}\big)^n$.

(1) Show that if $0 \leqslant a < b$, then
$$\frac{b^{n+1} - a^{n+1}}{b - a} < (n+1)b^n\,.$$

(2) Deduce that $b^n\big[(n+1)a - nb\big] < a^{n+1}$.

(3) Use $a = 1 + \dfrac{1}{n+1}$ and $b = 1 + \dfrac{1}{n}$ in (2) to show that $\{a_n\}_{n=1}^{\infty}$ is (strictly) increasing.

(4) Use $a = 1$ and $b = 1 + \dfrac{1}{2n}$ in (2) to show that $a_{2n} < 4$.

(5) Use (3) and (4) to show that $a_n < 4$.

(6) Deduce that $\{a_n\}_{n=1}^{\infty}$ converges.

Problem 8. Let a, b be positive real numbers, $a > b$. Let two sequence $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be given by the recursive relation

$$a_{n+1} = \frac{a_n + b_n}{2}\,,\ b_{n+1} = \sqrt{a_n b_n} \quad \forall\, n \in \mathbb{N}\,, \qquad a_1 = \frac{a+b}{2}\,,\ b_1 = \sqrt{ab}\,.$$

Complete the following.

(1) Show (by induction) that $a_n > a_{n+1} > b_{n+1} > b_n$ for all $n \in \mathbb{N}$.

(2) Deduce that $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ both converges.

(3) Show that $\lim\limits_{n\to\infty} a_n$ and $\lim\limits_{n\to\infty} b_n$ both exist and are identical.

Problem 9. Let $\{a_n\}_{n=0}^{\infty}$ be a sequence of real number defined by the recursive relation

$$a_{n+1} = \frac{1}{2 + a_n} \quad \forall\, n \geqslant 0\,, \qquad a_0 = \frac{1}{2}\,.$$

Complete the following.

(1) Show that the sequence $\{a_{2n}\}_{n=0}^{\infty}$ is a decreasing sequence; that is, $a_{2n+2} \leqslant a_{2n}$ for all $n \in \mathbb{N} \cup \{0\}$.

(2) Show that the sequence $\{a_{2n+1}\}_{n=0}^{\infty}$ is an increasing sequence; that is, $a_{2n+3} \geqslant a_{2n+1}$ for all $n \in \mathbb{N} \cup \{0\}$.

(3) Show that $a_{2k+1} \leqslant a_{2\ell}$ for all $k, \ell \in \mathbb{N} \cup \{0\}$.

(4) Show that the two sequences $\{a_{2n}\}_{n=0}^{\infty}$ and $\{a_{2n+1}\}_{n=0}^{\infty}$ converges to the same limit.

(5) Show that $\{a_n\}_{n=0}^{\infty}$ converges.

Problem 10. The Fibonacci sequence $\{f_n\}_{n=1}^{\infty}$ is a sequence defined recursively by

$$f_1 = 1\,,\quad f_2 = 1 \qquad \text{and} \qquad f_{n+2} = f_{n+1} + f_n \quad \forall\, n \in \mathbb{N}\,.$$

Show the following.

(1) $\dfrac{1}{f_{n-1}f_{n+1}} = \dfrac{1}{f_{n-1}f_n} - \dfrac{1}{f_n f_{n+1}}$ for all $n \geqslant 2$.

(2) $\sum\limits_{n=2}^{\infty} \dfrac{1}{f_{n-1}f_{n+1}} = 1$.

(3) $\sum\limits_{n=2}^{\infty} \dfrac{f_n}{f_{n-1}f_{n+1}} = 2$.

Problem 11. Consider the series $\sum\limits_{n=1}^{\infty} \dfrac{n}{(n+1)!}$.

(1) Find the partial sum S_1, S_2, S_3 and S_4. Do you recognize the denominators? Use the pattern to guess a formula for S_n.

(2) Prove your guess by induction.

(3) Show that the given series is convergent, and find the sum.