Calculus 微積分

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Chapter 4

Integration

- **The Σ notation**: The sum of n-terms $a_1, a_2, \cdots, a_n$ is written as $\sum\limits_{i=1}^{n} a_i$. In other words,

$$\sum_{i=1}^{n} a_i = a_1 + a_2 + \cdots + a_n\,.$$

Here i is called the index of summation, a_i is the i-th terms of the sum. We note that i in the sum $\sum\limits_{i=1}^{n} a_i$ is a dummy index which can be replaced by other indices such as j, k, and etc. Therefore, $\sum\limits_{i=1}^{n} a_i = \sum\limits_{j=1}^{n} a_j = \sum\limits_{k=1}^{n} a_k$, and so on.

- **Basic properties of sums**: $\displaystyle\sum_{i=1}^{n}(ca_i + b_i) = c\sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$.

Theorem 4.1: Summation Formula

1. $\sum\limits_{i=1}^{n} c = cn$ if c is a constant;
2. $\sum\limits_{i=1}^{n} i = \dfrac{n(n+1)}{2}$;
3. $\sum\limits_{i=1}^{n} i^2 = \dfrac{n(n+1)(2n+1)}{6}$;
4. $\sum\limits_{i=1}^{n} i^3 = \dfrac{n^2(n+1)^2}{4}$.

4.1 The Area under the Graph of a Non-negative Continuous Function

Let $f : [a, b] \to \mathbb{R}$ be a non-negative continuous function, and R be the region enclosed by the graph of the function f, the x-axis and straight lines $x = a$ and $x = b$. We consider computing $\mathcal{A}(\mathrm{R})$, the area of R. Generally speaking, since the graph of $y = f(x)$ is in general not a straight line, the computation of $\mathcal{A}(\mathrm{R})$ is not straight-forward. How do we compute the area $\mathcal{A}(\mathrm{R})$?

Partition $[a,b]$ into n sub-intervals with equal length, and let $\Delta x = \dfrac{b-a}{n}$, $x_i = a + i\Delta x$. By the Extreme Value Theorem, for each $1 \leqslant i \leqslant n$ f attains its maximum and minimum on $[x_{i-1}, x_i]$; thus for $1 \leqslant i \leqslant n$, there exist $M_i, m_i \in [x_{i-1}, x_i]$ such that

$$f(M_i) = \text{the maximum of } f \text{ on } [x_{i-1}, x_i]$$

and

$$f(m_i) = \text{the minimum of } f \text{ on } [x_{i-1}, x_i].$$

The sum $S(n) \equiv \sum\limits_{i=1}^{n} f(M_i)\Delta x$ is called the upper sum of f for the partition $\{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$, and $s(n) \equiv \sum\limits_{i=1}^{n} f(m_i)\Delta x$ is called the lower sum of f for the partition $\{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$. By the definition of the upper sum and lower sum, we find that for each $n \in \mathbb{N}$,

$$\sum_{i=1}^{n} f(m_i)\Delta x \leqslant \mathcal{A}(\mathrm{R}) \leqslant \sum_{i=1}^{n} f(M_i)\Delta x\,.$$

If the limits of the both sides exist and are identical as Δx approaches 0 (which is the same as n approaches infinity), by the Squeeze Theorem we can conclude that $\mathcal{A}(\mathrm{R})$ is the same as the limit.

Example 4.2. Let $f(x) = x^2$, and R be the region enclosed by the graph of $y = f(x)$, the X-axis, and the straight lines $x = a$ and $x = b$, where we assume that $0 \leqslant a < b$. Then the lower sum is obtained by the "left end-point rule" approximation of $\mathcal{A}(\mathrm{R})$

$$\sum_{i=1}^{n} \Big(a + \frac{(i-1)(b-a)}{n}\Big)^2 \frac{b-a}{n}$$

and the upper sum is obtained by the "right end-point rule" approximation

$$\sum_{i=1}^{n} \Big(a + \frac{i(b-a)}{n}\Big)^2 \frac{b-a}{n}\,.$$

By Theorem 4.1,

$$\begin{aligned}
\sum_{i=1}^{n} \Big(a + \frac{i(b-a)}{n}\Big)^2 \frac{b-a}{n} &= \sum_{i=1}^{n} \Big[a^2 + \frac{2a(b-a)i}{n} + \frac{a^2(b-a)^2 i^2}{n^2}\Big] \frac{b-a}{n} \\
&= a^2(b-a) + \frac{a(b-a)^2 n(n+1)}{n^2} + \frac{a^2(b-a)^3}{n^3}\frac{n(n+1)(2n+1)}{6} \\
&= a^2(b-a) + a(b-a)^2\Big(1 + \frac{1}{n}\Big) + \frac{a^2(b-a)^3}{6}\Big(1 + \frac{1}{n}\Big)\Big(2 + \frac{1}{n}\Big)\,.
\end{aligned}$$

Letting $n \to \infty$, we find that

$$\lim_{n\to\infty} \sum_{i=1}^{n} \Big(a + \frac{i(b-a)}{n}\Big)^2 \frac{b-a}{n} = a^2(b-a) + a(b-a)^2 + \frac{a^2(b-a)^3}{3} = \frac{b^3-a^3}{3}\,.$$

Similarly,

$$\begin{aligned}\sum_{i=1}^{n} \Big(a + \frac{(i-1)(b-a)}{n}\Big)^2 \frac{b-a}{n} &= \frac{a^2(b-a)}{n} + \sum_{i=1}^{n} \Big(a + \frac{i(b-a)}{n}\Big)^2 \frac{b-a}{n} - \frac{b^2(b-a)}{n} \\ &= a^2(b-a) + \frac{a(b-a)^2 n(n+1)}{n^2} + \frac{a^2(b-a)^3}{n^3}\frac{n(n+1)(2n+1)}{6} + \frac{(a^2-b^2)(b-a)}{n}\,;\end{aligned}$$

thus

$$\lim_{n\to\infty} \sum_{i=1}^{n} \Big(a + \frac{(i-1)(b-a)}{n}\Big)^2 \frac{b-a}{n} = \frac{b^3-a^3}{3}\,.$$

Therefore, $\mathcal{A}(\mathrm{R}) = \dfrac{b^3-a^3}{3}$.

Remark 4.3. Let R_1 be the region enclosed by $f(x) = x^2$, the x-axis and $x = a$, the R_2 be the region enclosed by $f(x) = x^2$, the x-axis and $x = b$, then intuitively $\mathcal{A}(\mathrm{R}) = \mathcal{A}(\mathrm{R}_2) - \mathcal{A}(\mathrm{R}_1)$ and this is true since $\mathcal{A}(\mathrm{R}_1) = \dfrac{a^3}{3}$ and $\mathcal{A}(\mathrm{R}_2) = \dfrac{b^3}{3}$.

If f is not continuous, then f might not attain its extrema on the interval $[x_{i-1}, x_i]$. In this case, it might be impossible to form the upper sum or the lower sum for a given partition. On the other hand, the left end-point rule $\sum\limits_{i=1}^{n} f(x_{i-1})\Delta x$ and the right end-point rule $\sum\limits_{i=1}^{n} f(x_i)\Delta x$ of approximating the area are always possible. We can even consider the "mid-point rule" approximation given by

$$\sum_{i=1}^{n} f\big(\frac{x_{i-1}+x_i}{2}\big)\Delta x$$

and consider the limit of the expression above as n approaches infinity.

4.2 Riemann Sums and Definite Integrals

In general, in order to find an approximation of $\mathcal{A}(\mathrm{R})$, the interval $[a, b]$ does not have to be divided into sub-intervals with equal length. Assume that $[a, b]$ are divided into n sub-intervals and the end-points of those sub-intervals are ordered as $a = x_0 < x_1 < x_2 < \cdots <$

$x_n = b$, here the collection of end-points $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ is called a ***partition*** of $[a, b]$. Then the "left end-point rule" approximation for the partition $\mathcal{P}$ is given by

$$\ell(\mathcal{P}) = \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1})$$

and the "right end-point rule" approximation for the partition $\mathcal{P}$ is given by

$$r(\mathcal{P}) = \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}),$$

and the limit process as $n \to \infty$ in the discussion above is replaced by the limit process as the norm of partition $\mathcal{P}$, denoted by $\|\mathcal{P}\|$ and defined by $\|\mathcal{P}\| \equiv \max\big\{x_i - x_{i-1} \,\big|\, 1 \leqslant i \leqslant n\big\}$, approaches 0. Before discussing what the limits above mean, let us look at the following examples.

Example 4.4. Consider the region bounded by the graph of $f(x) = \sqrt{x}$ and the x-axis for $0 \leqslant x \leqslant 1$. Let $x_i = \dfrac{i^2}{n^2}$ and $\mathcal{P} = \{x_0 = 0 < x_1 < \cdots < x_n = 1\}$. We note that

$$\|\mathcal{P}\| = \max\Big\{\frac{i^2 - (i-1)^2}{n^2} \,\Big|\, 1 \leqslant i \leqslant n\Big\} = \max\Big\{\frac{2i-1}{n^2} \,\Big|\, 1 \leqslant i \leqslant n\Big\} = \frac{2n-1}{n^2}$$

thus $\|\mathcal{P}\| \to 0$ is equivalent to that $n \to \infty$.

Using the right end-point rule (which is the same as the upper sum),

$$\begin{aligned} S(\mathcal{P}) &= \sum_{i=1}^{n} f(x_i)(x_i - x_{i-1}) = \sum_{i=1}^{n} \frac{i}{n}\frac{2i-1}{n^2} = \frac{1}{n^3}\sum_{i=1}^{n}(2i^2 - i) \\ &= \frac{1}{n^3}\Big[\frac{n(n+1)(2n+1)}{3} - \frac{n(n+1)}{2}\Big] \\ &= \frac{1}{3}\Big(1 + \frac{1}{n}\Big)\Big(2 + \frac{1}{n}\Big) - \frac{1}{2n}\Big(1 + \frac{1}{n}\Big); \end{aligned}$$

thus

$$\lim_{\|\mathcal{P}\| \to 0} S(\mathcal{P}) = \lim_{n \to \infty}\Big[\frac{1}{3}\Big(1 + \frac{1}{n}\Big)\Big(2 + \frac{1}{n}\Big) - \frac{1}{2n}\Big(1 + \frac{1}{n}\Big)\Big] = \frac{2}{3}.$$

Using the left end-point rule (which is the same as the lower sum),

$$\begin{aligned} s(\mathcal{P}) &= \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1}) = \sum_{i=1}^{n} \frac{i-1}{n}\frac{2i-1}{n^2} = \frac{1}{n^3}\sum_{i=1}^{n}(2i^2 - 3i + 1) \\ &= \frac{1}{n^3}\Big[\frac{n(n+1)(2n+1)}{3} - \frac{3n(n+1)}{2} + n\Big] \\ &= \frac{1}{3}\Big(1 + \frac{1}{n}\Big)\Big(2 + \frac{1}{n}\Big) - \frac{3}{2n}\Big(1 + \frac{1}{n}\Big) + \frac{1}{n^2}; \end{aligned}$$

thus

$$\lim_{\|\mathcal{P}\|\to 0} s(\mathcal{P}) = \lim_{n\to\infty} \Big[\frac{1}{3}\Big(1+\frac{1}{n}\Big)\Big(2+\frac{1}{n}\Big) - \frac{3}{2n}\Big(1+\frac{1}{n}\Big) + \frac{1}{n^2}\Big] = \frac{2}{3}\,.$$

Therefore, the area of the region of interest is $\frac{2}{3}$.

Example 4.5. In this example we use a different approach to compute $\mathcal{A}(\mathrm{R})$ in Example 4.2. Assume that $0 < a < b$. Let $r = \big(\frac{b}{a}\big)^{\frac{1}{n}}$, $x_i = ar^i$, and $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$.
Claim: If $c > 1$, then $c^{\frac{1}{n}} = 1$ as n approaches infinity.
Proof of the claim: If $c > 1$, then $c^{\frac{1}{n}} > 1$. Let $y_n = c^{\frac{1}{n}} - 1$. Then $c = (1 + y_n)^n \geqslant 1 + ny_n$ which implies that $0 < y_n \leqslant \frac{c-1}{n}$ for all $n \in \mathbb{N}$. By the Squeeze Theorem, $c^{\frac{1}{n}} \to 1$ as $n \to \infty$. □

Note that the claim above implies that $r \to 1$ as $n \to \infty$. Moreover, $x_i - x_{i-1} = a(r^i - r^{i-1}) = ar^{i-1}(r-1)$; thus

$$0 < a(r-1) = x_1 - x_0 \leqslant \|\mathcal{P}\| = x_n - x_{n-1} = ar^{n-1}(r-1) < b(r-1)\,.$$

Therefore, $\|\mathcal{P}\| \to 0$ is equivalent to that $n \to \infty$.

Using the "left end-point rule" approximation of the area,

$$\begin{aligned}\mathcal{A}(\mathrm{R}) &= \lim_{n\to\infty}\sum_{i=1}^n x_{i-1}^2(x_i - x_{i-1}) = \lim_{n\to\infty}\sum_{i=1}^n a^2r^{2(i-1)}ar^{i-1}(r-1) = a^3\lim_{n\to\infty}(r-1)\sum_{i=1}^n r^{3(i-1)} \\ &= a^3\lim_{n\to\infty}(r-1)\frac{r^{3n}-1}{r^3-1} = a^3\lim_{n\to\infty}\frac{\frac{b^3}{a^3}-1}{r^2+r+1} = \frac{b^3-a^3}{3}\,.\end{aligned}$$

Similarly, when applying the "right end-point rule" approximation, we obtain that

$$\lim_{n\to\infty}\sum_{i=1}^n x_i^2(x_i - x_{i-1}) = a^3\lim_{n\to\infty}(r-1)\sum_{i=1}^n r^{3i} = a^3\lim_{n\to\infty}(r-1)\frac{r^{3n+3}-r^3}{r^3-1} = \frac{b^3-a^3}{3}\,.$$

This also gives the area of the region R.

To compute an approximated value of $\mathcal{A}(\mathrm{R})$, there is no reason for evaluating the function at the left end-points or the right end-points like what we have discussed above. For example, we can also consider the "mid-point rule"

$$m(\mathcal{P}) = \sum_{i=1}^n f\Big(\frac{x_i + x_{i-1}}{2}\Big)(x_i - x_{i-1})$$

to approximate the value of $\mathcal{A}(\mathrm{R})$, and compute the limit of the sum above as $\|\mathcal{P}\|$ approaches 0 in order to obtain $\mathcal{A}(\mathrm{R})$. In fact, we should be able to consider any point $c_i \in [x_{i-1}, x_i]$ and consider the limit of the sum

$$\lim_{\|\mathcal{P}\|\to 0} \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1})$$

if the region R does have area.

Now let us forget about the concept of the area. For a general function $f : [a, b] \to \mathbb{R}$, we can also consider the limit above as $\|\mathcal{P}\|$ approaches 0, if the limit exists. The discussion above motivates the following definitions.

Definition 4.6: Partition of Intervals and Riemann Sums

A finite set $\mathcal{P} = \{x_0, x_1, \cdots, x_n\}$ is said to be a partition of the closed interval $[a, b]$ if $a = x_0 < x_1 < \cdots < x_n = b$. Such a partition $\mathcal{P}$ is usually denoted by $\{a = x_0 < x_1 < \cdots < x_n\}$. The norm of $\mathcal{P}$, denoted by $\|\mathcal{P}\|$, is the number $\max\{x_i - x_{i-1} \,|\, 1 \leqslant i \leqslant n\}$; that is,

$$\|\mathcal{P}\| \equiv \max\{x_i - x_{i-1} \,\big|\, 1 \leqslant i \leqslant n\}.$$

A partition $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ is called regular if $x_i - x_{i-1} = \|\mathcal{P}\|$ for all $1 \leqslant i \leqslant n$.

Let $f : [a, b] \to \mathbb{R}$ be a function. A Riemann sum of f for the the partition $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ of $[a, b]$ is a sum which takes the form

$$\sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}),$$

where the set $\Xi = \{c_0, c_1, \cdots, c_{n-1}\}$ satisfies that $x_{i-1} \leqslant c_i \leqslant x_i$ for each $1 \leqslant i \leqslant n$.

Definition 4.7: Riemann Integrals - 黎曼積分

Let $f : [a, b] \to \mathbb{R}$ be a function. f is said to be Riemann integrable on $[a, b]$ if there exists a real number A such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\mathcal{P}$ is partition of $[a, b]$ satisfying $\|\mathcal{P}\| < \delta$, then any Riemann sums for the partition $\mathcal{P}$ belongs to the interval $(A - \varepsilon, A + \varepsilon)$. Such a number A (is unique and) is called the Riemann integral of f on $[a, b]$ and is denoted by $\displaystyle\int_{[a,b]} f(x)\,dx$.

Remark 4.8. For conventional reason, the Riemann integral of f over the interval with left end-point a and right-end point b is written as $\displaystyle\int_a^b f(x)\,dx$, and is called the definite integral of f from a to b. The function f sometimes is called the integrand of the integral.

We also note that here in the representation of the integral, x is a dummy variable; that is, we can use any symbol to denote the independent variable; thus

$$\int_a^b f(x)\,dx = \int_a^b f(t)\,dt = \int_a^b f(u)\,du$$

and etc.

The following example shows that no all functions are Riemann integrable.

Example 4.9. Consider the Dirichlet function

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational}\,, \\ 1 & \text{if } x \text{ is irrational}\,, \end{cases}$$

on the interval $[1, 2]$. By partitioning $[1, 2]$ into n sub-intervals with equal length, the Riemann sum given by the right end-point rule is always zero since the right end-point of each sub-interval is rational. On the other hand, by partitioning $[1, 2]$ into n sub-intervals using geometric sequence $1, r, r^2, \cdots, r^{n-1}, 2$, where $r = 2^{\frac{1}{n}}$, by the fact that $r^i \notin \mathbb{Q}$ for each $1 \leqslant i \leqslant n-1$ the Riemann sum of f for this partition given by the right end-point rule is

$$\begin{aligned} \sum_{i=1}^{n} f(r^i)(r^i - r^{i-1}) &= \sum_{i=1}^{n-1} (r^i - r^{i-1}) = r^1 - r^0 + r^2 - r^1 + \cdots + r^{n-1} - r^{n-2} \\ &= r^{n-1} - r^0 = \frac{2}{r} - 1 \end{aligned}$$

which approaches 1 as r approaches 1. Therefore, f is not integrable on $[1, 2]$ since there are two possible limits of Riemann sums which means that the Riemann sums cannot concentrate around any firxed real number.

Theorem 4.10

If $f : [a, b] \to \mathbb{R}$ is continuous, then f is Riemann integrable on $[a, b]$.

Example 4.11. In this example we compute $\displaystyle\int_a^b x^q\,dx$ when $q \neq -1$ is a rational number and $0 < a < b$. Since $f(x) = x^q$ is continuous on $[a, b]$, by Theorem 4.10 to find the integral it suffices to find the limit of the Riemann sum given by the left end-point rule as $\|\mathcal{P}\|$ approaches 0.

We follow the idea in Example 4.5. Let $r = \left(\frac{b}{a}\right)^{\frac{1}{n}}$ and $x_i = ar^i$, as well as the partition $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$. Then the Riemann sum of f for the partition $\mathcal{P}$ given by left end-point rule is

$$\begin{aligned} L(\mathcal{P}) &= \sum_{i=1}^{n} (ar^{i-1})^q (ar^i - ar^{i-1}) = a^{q+1}(r-1)\sum_{i=1}^{n} r^{(i-1)(q+1)} = a^{q+1}(r-1)\frac{r^{n(q+1)}-1}{r^{q+1}-1} \\ &= \frac{r-1}{r^{q+1}-1}\big(b^{q+1} - a^{q+1}\big)\,. \end{aligned}$$

Since $\frac{d}{dr}\Big|_{r=1} r^{q+1} = (q+1)$, we have

$$\lim_{r\to 1} \frac{r^{q+1}-1}{r-1} = \frac{d}{dr}\Big|_{r=1} r^{q+1} = q+1\,;$$

thus by the fact that $r \to 1$ as $n \to \infty$ (or $\|\mathcal{P}\| \to 0$), we find that

$$\lim_{\|\mathcal{P}\|\to 0} L(\mathcal{P}) = \lim_{\|\mathcal{P}\|\to 0} L(\mathcal{P}) = \frac{b^{q+1} - a^{q+1}}{q+1}\,.$$

Therefore, $\displaystyle\int_a^b x^q\,dx = \frac{b^{q+1}-a^{q+1}}{q+1}$ if $q \neq 1$ is a rational number and $0 < a < b$.

Example 4.12. Since the sine function is continuous on any closed interval $[a,b]$, to find $\displaystyle\int_a^b \sin x\,dx$ we can partition $[a,b]$ into sub-intervals with equal length, use the right end-point rule to find an approximated value of the integral, and finally find the integral by passing the number of sub-intervals to the limit.

Let $\Delta x = \frac{b-a}{n}$ and $x_i = a + i\Delta x$. The right end-point rule gives the approximation

$$\sum_{i=1}^{n} \sin x_i \Delta x = \sum_{i=1}^{n} \sin(a + i\Delta x)\Delta x = \Delta x \sum_{i=1}^{n} \sin(a + i\Delta x)$$

of the integral.

Using the sum and difference formula, we find that

$$\cos\Big[a + \big(i - \frac{1}{2}\big)\Delta x\Big] - \cos\Big[a + \big(i + \frac{1}{2}\big)\Delta x\Big] = 2\sin(a + i\Delta x)\sin\frac{\Delta x}{2}\,;$$

thus if $\sin\frac{\Delta x}{2} \neq 0$,

$$\sum_{i=1}^{n} \sin(a+i\Delta x) = \frac{1}{2\sin\frac{\Delta x}{2}}\Big[\Big(\cos\big(a+\frac{1}{2}\Delta x\big) - \cos\big(a+\frac{3}{2}\Delta x\big)\Big) + \Big(\cos\big(a+\frac{3}{2}\Delta x\big) - \cos\big(a+\frac{5}{2}\Delta x\big)\Big) + \cdots + \cos\big[a+(n-\frac{1}{2})\Delta x\big] - \cos\big[a+(n+\frac{1}{2})\Delta x\big]\Big]$$

which, by the fact that $a + \big(n+\frac{1}{2}\Delta x\big) = b + \frac{1}{2}\Delta x$, implies that

$$\sum_{i=1}^{n} \sin x_i \Delta x = \frac{\frac{\Delta x}{2}}{\sin\frac{\Delta x}{2}}\Big[\cos\big(a+\frac{1}{2}\Delta x\big) - \cos\big(b+\frac{1}{2}\Delta x\big)\Big].$$

By the fact that $\lim\limits_{x\to 0}\frac{\sin x}{x} = 1$ and the continuity of the cosine function, we conclude that

$$\int_a^b \sin x\,dx = \lim_{n\to\infty}\sum_{i=1}^{n} \sin x_i \Delta x = \cos a - \cos b\,.$$

Theorem 4.13

Let $f : [a,b] \to \mathbb{R}$ be a non-negative and continuous function. The area of the region enclosed by the graph of f, the x-axis, and the vertical lines $x = a$ and $x = b$ is $\int_a^b f(x)\,dx$.

Example 4.14. In this example we use the integral notation to denote the areas of some common geometric figures (without really doing computations):

1. $\int_{-2}^{2}\sqrt{4-x^2}\,dx = 2\pi\,;$ 2. $\int_{-1}^{1}\sqrt{4-x^2}\,dx = \frac{2\pi}{3} + \sqrt{3}\,;$ 3. $\int_{-1}^{\sqrt{3}}\sqrt{4-x^2}\,dx = \pi + \sqrt{3}.$

4.2.1 Properties of Definite Integrals

Definition 4.15

1. If f is defined at $x = a$, then $\int_a^a f(x)\,dx = 0$.
2. If f is integrable on $[a,b]$, then $\int_b^a f(x)\,dx = -\int_a^b f(x)\,dx = -\int_{[a,b]} f(x)\,dx$.

Remark 4.16. By the definition above, if f is Riemann integrable on $[a, b]$, $\displaystyle\int_b^a f(x)\,dx$ is the limit of the sum

$$\sum_{i=1}^{n} f(x_i)(x_i - x_{i-}) \quad \textit{and} \quad \sum_{i=1}^{n} f(x_{i-1})(x_i - x_{i-1})$$

as $\max\left\{|x_i - x_{i-1}| \,\middle|\, 1 \leqslant i \leqslant n\right\} \to 0$, where $x_0 = b > x_1 > x_2 > \cdots > x_n = a$.

Theorem 4.17

If f is Riemann integrable on the three closed intervals determined by a, b and c, then

$$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx\,.$$

Theorem 4.18

Let $f, g : [a, b] \to \mathbb{R}$ be Riemann integrable on $[a, b]$ and k be a constant. Then the function $kf \pm g$ are Riemann integrable on $[a, b]$, and

$$\int_a^b (kf \pm g)(x)\,dx = k\int_a^b f(x)\,dx \pm \int_a^b g(x)\,dx\,.$$

Theorem 4.19

If f is non-negative and Riemann integrable on $[a, b]$, then $\displaystyle\int_a^b f(x)\,dx \geqslant 0$.

Corollary 4.20

If f, g are Riemann integrable on $[a, b]$ and $f(x) \leqslant g(x)$ for all $a \leqslant x \leqslant b$, then

$$\int_a^b f(x)\,dx \leqslant \int_a^b g(x)\,dx\,.$$

Theorem 4.21

If f is Riemann integrable on $[a, b]$, then $|f|$ is Riemann integrable on $[a, b]$ and

$$\left|\int_a^b f(x)\,dx\right| \leqslant \int_a^b \big|f(x)\big|\,dx\,.$$

Theorem 4.22: 可積必有界

Let $f : [a, b] \to \mathbb{R}$ be a function. If f is Riemann integrable on $[a, b]$, then f is bounded on $[a, b]$; that is, there exists $M > 0$ such that

$$\big|f(x)\big| \leqslant M \qquad \text{whenever} \quad x \in [a, b] \,.$$

Proof. Let f be Riemann integrable on $[a, b]$. Then there exists $A \in \mathbb{R}$ and $\delta > 0$ such that if $\mathcal{P}$ is a partition of $[a, b]$ satisfying $\|\mathcal{P}\| < \delta$, then any Riemann sum of f for $\mathcal{P}$ belongs to $(A - 1, A + 1)$. Choose $n \in \mathbb{N}$ so that $\dfrac{b-a}{n} < \delta$. Then the regular partition $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$, where $x_i = a + \dfrac{b-a}{n} i$, satisfies $\|\mathcal{P}\| < \delta$.

Suppose the contrary that f is not bounded. Then there exists $x^* \in [a, b]$ such that

$$\big|f(x^*)\big| > \frac{n(|A| + 1)}{b-a} + \sum_{i=1}^{n} \big|f(x_i)\big| \,.$$

Suppose that $x^* \in [x_{k-1}, x_k]$. By the fact that $\sum\limits_{\substack{i=1 \\ i \neq k}}^{n} f(x_i)(x_i - x_{i-1}) + f(x^*)(x_k - x_{k-1})$ is a Riemann sum of f for $\mathcal{P}$, we have

$$A - 1 < \sum_{\substack{i=1 \\ i \neq k}}^{n} f(x_i)(x_i - x_{i-1}) + f(x^*)(x_k - x_{k-1}) < A + 1 \,.$$

Since $x_i - x_{i-1} = \dfrac{b-1}{n}$ for all $1 \leqslant i \leqslant n$, the inequality above shows that

$$\frac{n(A-1)}{b-a} - \sum_{\substack{i=1 \\ i \neq k}}^{n} f(x_i) < f(x^*) < \frac{n(A+1)}{b-a} - \sum_{\substack{i=1 \\ i \neq k}}^{n} f(x_i)$$

and the triangle inequality further implies that

$$-\Big[\frac{n(|A|+1)}{b-a} + \sum_{\substack{i=1 \\ i \neq k}}^{n} \big|f(x_i)\big|\Big] < f(x^*) < \frac{n(|A|+1)}{b-a} + \sum_{\substack{i=1 \\ i \neq k}}^{n} \big|f(x_i)\big| \,.$$

Therefore, we conclude that

$$\big|f(x^*)\big| < \frac{n(|A|+1)}{b-a} + \sum_{\substack{i=1 \\ i \neq k}}^{n} \big|f(x_i)\big| \leqslant \frac{n(|A|+1)}{b-a} + \sum_{i=1}^{n} \big|f(x_i)\big| \,,$$

a contradiction. $\square$

Example 4.23. Let $f : [0, 1] \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} \dfrac{1}{x} & \text{if } x \in (0, 1]\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$

Then f has only one discontinuity in $[0, 1]$ but f is not Riemann integrable on $[0, 1]$ since f is not bounded.

4.3 The Fundamental Theorem of Calculus

In this section, we develop a theory which shows a systematic way of finding integrals if the integrand is a continuous function.

Definition 4.24

A function F is an anti-derivative of f on an interval I if $F'(x) = f(x)$ for all x in I.

Theorem 4.25

If F is an anti-derivative of f on an interval I, then G is an anti-derivative of f on the interval I if and only if G is of the form $G(x) = F(x) + C$ for all x in I, where C is a constant. (導函數相同的函數相差一常數)

Proof. It suffices to show the "$\Rightarrow$" (only if) direction. Suppose that $F' = G' = f$ on I. Then the function $h = F - G$ satisfies $h'(x) = 0$ for all $x \in I$. By the mean value theorem, for any $a, b \in I$ with $a \neq b$, there exists c in between a and b such that

$$h(b) - h(a) = h'(c)(b - a)\,.$$

Since $h'(x) = 0$ for all $x \in I$, $h(a) = h(b)$ for all $a, b \in I$; thus h is a constant function. $\square$

Theorem 4.26: Mean Value Theorem for Integrals - 積分均值定理

Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then there exists $c \in [a, b]$ such that

$$\int_a^b f(x)\,dx = f(c)(b - a)\,.$$

Proof. By the Extreme Value Theorem, f has a maximum and a minimum on $[a, b]$. Let $M = f(x_1)$ and $m = f(x_2)$, where $x_1, x_2 \in [a, b]$, denote the maximum and minimum of f

on $[a,b]$, respectively. Then $m \leqslant f(x) \leqslant M$ for all $x \in [a,b]$; thus Corollary 4.20 implies that

$$m(b-a) = \int_a^b m\,dx \leqslant \int_a^b f(x)\,dx \leqslant \int_a^b M\,dx = M(b-a)\,.$$

Therefore, the number $\dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx \in [m,M]$. By the Intermidiate Value Theorem, there exists c in between x_1 and x_2 such that $f(c) = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx$. □

Theorem 4.27: Fundamental Theorem of Calculus - 微積分基本定理

Let $f : [a,b] \to \mathbb{R}$ be a continuous function, and F be an anti-derivative of f on $[a,b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a)\,.$$

Moreover, if $G(x) = \displaystyle\int_a^x f(t)\,dt$ for $x \in [a,b]$, then G is an anti-derivative of f.

We note that for $x \in [a,b]$, f is continuous on $[a,x]$; thus f is Riemann integrable on $[a,x]$ which shows that $G(x) = \displaystyle\int_a^x f(t)\,dt$ is well-defined.

Proof of the Fundamental Theorem of Calculus. Note that for $h \neq 0$ such that $x+h \in [a,b]$, we have

$$\frac{G(x+h)-G(x)}{h} = \frac{1}{h}\Big[\int_a^{x+h} f(t)\,dt - \int_a^x f(t)\,dt\Big] = \frac{1}{h}\int_x^{x+h} f(t)\,dt\,.$$

By the Mean Value Theorem for Integrals, there exists $c = c(h)$ in between x and $x+h$ such that $\dfrac{1}{h}\displaystyle\int_x^{x+h} f(t)\,dt = f(c)$. Since f is continuous on $[a,b]$, $\lim\limits_{h\to 0} f(c) = \lim\limits_{c\to x} f(c) = f(x)$; thus

$$\lim_{h\to 0}\frac{G(x+h)-G(x)}{h} = \lim_{h\to 0}\frac{1}{h}\int_x^{x+h} f(t)\,dt = \lim_{h\to 0} f(c) = f(x)$$

which shows that G is an anti-derivative of f on $[a,b]$.

By Theorem 4.25, $G(x) = F(x) + C$ for all $x \in [a,b]$. By the fact that $G(a) = 0$, $C = -F(a)$; thus

$$\int_a^b f(x)\,dx = G(b) = F(b) - F(a)$$

which concludes the theorem. □

Example 4.28. Since an anti-derivative of the function $y = x^q$, where $q \neq -1$ is a rational number, is $y = \dfrac{x^{q+1}}{q+1}$, we find that

$$\int_a^b x^q\,dx = \frac{x^{q+1}}{q+1}\Big|_{x=b} - \frac{x^{q+1}}{q+1}\Big|_{x=a} = \frac{b^{q+1} - a^{q+1}}{q+1}\,.$$

Example 4.29. Since an anti-derivative of the sine function is negative of cosine, we find that

$$\int_a^b \sin x\,dx = (-\cos)(b) - (-\cos)(b) = \cos b - \cos a\,.$$

Example 4.30. Find $\dfrac{d}{dx}\displaystyle\int_0^{\sqrt{x}} \sin^{100} t\,dt$ for $x > 0$.

Let $F(x) = \displaystyle\int_0^x \sin^{100} t\,dt$. Then by the chain rule,

$$\frac{d}{dx}F(\sqrt{x}) = F'(\sqrt{x})\frac{d}{dx}\sqrt{x} = \frac{1}{2\sqrt{x}}F'(\sqrt{x})\,.$$

By the Fundamental Theorem of Calculus, $F'(x) = \sin^{100} x$; thus

$$\frac{d}{dx}\int_0^{\sqrt{x}} \sin^{100} t\,dt = \frac{d}{dx}F(\sqrt{x}) = \frac{\sin^{100}\sqrt{x}}{2\sqrt{x}}\,.$$

Theorem 4.31

Let $f : [a,b] \to \mathbb{R}$ be continuous and f is differentiable on (a,b). If f' is Riemann integrable on $[a,b]$, then

$$\int_a^b f'(x)\,dx = f(b) - f(a)\,.$$

Proof. Let $\varepsilon > 0$ be given, and define $A = \displaystyle\int_a^b f'(x)\,dx$. By the definition of the integrability there exists $\delta > 0$ such that if $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ is a partition of $[a,b]$ satisfying $\|\mathcal{P}\| < \delta$, then any Riemann sums of f for $\mathcal{P}$ belongs to the interval $(A-\varepsilon, A+\varepsilon)$.

Let $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ be a partition of $[a,b]$ satisfying that $\|\mathcal{P}\| < \delta$. Then by the mean value theorem, for each $1 \leqslant i \leqslant n$ there exists $x_{i-1} < c < x_i$ such that $f(x_i) - f(x_{i-1}) = f'(c_i)(x_i - x_{i-1})$. Since

$$\sum_{i=1}^n f'(c_i)(x_i - x_{i-1})$$

is a Riemann sum of f for $\mathcal{P}$, we must have

$$\Big|\sum_{i=1}^{n} f'(c_i)(x_i - x_{i-1}) - A\Big| < \varepsilon\,.$$

On the other hand, by the fact that

$$\begin{aligned}\sum_{i=1}^{n} f'(c_i)(x_i - x_{i-1}) &= \sum_{i=1}^{n}\big[f(x_i) - f(x_{i-1})\big] \\ &= f(x_1) - f(x_0) + f(x_2) - f(x_1) + \cdots + f(x_n) - f(x_{n-1}) \\ &= f(x_n) - f(x_0) = f(b) - f(a)\,,\end{aligned}$$

we conclude that

$$\Big|f(b) - f(a) - \int_a^b f'(x)\,dx\Big| < \varepsilon\,.$$

Since $\varepsilon > 0$ is chosen arbitrarily, we find that $\displaystyle\int_a^b f'(x)\,dx = f(b) - f(a)$. □

Remark 4.32. If f' is continuous on $[a,b]$, then the theorem above is simply a direct consequence of the Fundamental Theorem of Calculus. The theorem above can be viewed as a generalization of the Fundamental Theorem of Calculus.

Theorem 4.27 and 4.31 can be combined as follows:

Theorem 4.33

Let $f : [a,b] \to \mathbb{R}$ be a Riemann integrable function and F be an anti-derivative of f on $[a,b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a)\,.$$

Moreover, if in addition f is continuous on $[a,b]$, then $G(x) = \displaystyle\int_a^x f(t)\,dt$ is differentiable on $[a,b]$ and

$$G'(x) = f(x) \qquad \text{for all } x \in [a,b]\,.$$

Definition 4.34

An anti-derivative of f, if exists, is denoted by $\displaystyle\int f(x)\,dx$, and sometimes is also called an indefinite integral of f.

- Basic Rules of Integration:

Differentiation Formula	Anti-derivative Formula
$\frac{d}{dx}C = 0$	$\int 0\,dx = C$
$\frac{d}{dx}x^r = rx^{r-1}$	$\int x^q\,dx = \frac{x^{q+1}}{q+1} + C \;\text{ if } q \neq -1$
$\frac{d}{dx}\sin x = \cos x$	$\int \cos x\,dx = \sin x + C$
$\frac{d}{dx}\cos x = -\sin x$	$\int \sin x\,dx = -\cos x + C$
$\frac{d}{dx}\tan x = \sec^2 x$	$\int \sec^2 x\,dx = \tan x + C$
$\frac{d}{dx}\sec x = \sec x \tan x$	$\int \sec x \tan x\,dx = \sec x + C$
$\frac{d}{dx}\big[kf(x) + g(x)\big] = kf'(x) + g'(x)$	$\int \big[kf'(x) + g'(x)\big]\,dx = kf(x) + g(x) + C$

4.4 Integration by Substitution - 變數變換

Suppose that $g : [a, b] \to \mathbb{R}$ is differentiable, and $f : \text{range}(g) \to \mathbb{R}$ is differentiable. Then the chain rule implies that $f \circ g$ is an anti-derivative of $(f' \circ g)g'$; thus provided that

1. $(f \circ g)'$ is Riemann integrable on $[a, b]$,
2. f' is Riemann integrable on the range of g,

then Theorem 4.31 implies that

$$\begin{aligned}\int_a^b f'\big(g(x)\big)g'(x)\,dx &= \int_a^b (f \circ g)'(x)\,dx = (f \circ g)(b) - (f \circ g)(a) \\ &= f\big(g(b)\big) - f\big(g(a)\big) = \int_{g(a)}^{g(b)} f'(u)\,du\,. \end{aligned} \tag{4.4.1}$$

Replacing f' by f in the identity above shows the following

Theorem 4.35

If the function $u = g(x)$ has a continuous derivative on the closed interval $[a, b]$, and f is continuous on the range of g, then

$$\int_a^b f\big(g(x)\big)g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du\,.$$

The anti-derivative version of Theorem 4.35 is stated as follows.

Theorem 4.36

Let g be a function with range I and f be a continuous function on I. If g is differentiable on its domain and F is an anti-derivative of f on I, then

$$\int f\big(g(x)\big)g'(x)\,dx = F\big(g(x)\big) + C$$

Letting $u = g(x)$ gives $du = g'(x)\,dx$ and

$$\int f(u)\,du = F(u) + C\,.$$

Example 4.37. Find $\displaystyle\int (x^2+1)^2(2x)\,dx$.

Let $u = x^2+1$. Then $du = 2xdx$; thus

$$\int (x^2+1)^2(2x)\,dx = \int u^2\,du = \frac{1}{3}u^3 + C = \frac{1}{3}(x^2+1)^3 + C\,.$$

Example 4.38. Find $\displaystyle\int \cos(5x)\,dx$.

Let $u = 5x$. Then $du = 5dx$; thus

$$\int \cos(5x)\,dx = \frac{1}{5}\int \cos u\,du = \frac{1}{5}\sin u + C = \frac{1}{5}\sin(5x) + C\,.$$

Example 4.39. Find $\displaystyle\int \sec^2 x(\tan x+3)\,dx$.

Let $u = \tan x$. Then $du = \sec^2 x dx$; thus

$$\int \sec^2 x(\tan x+3)\,dx = \int (u+3)\,du = \frac{1}{2}u^2 + 3u + C = \frac{1}{2}\tan^2 x + 3\tan x + C\,.$$

On the other hand, let $v = \tan x + 3$. Then $dv = \sec^2 x\,dx$; thus

$$\begin{aligned}\int \sec^2 x(\tan x+3)\,dx &= \int v\,dv = \frac{1}{2}v^2 + C = \frac{1}{2}(\tan x+3)^2 + C\\ &= \frac{1}{2}\tan^2 x + 3\tan x + \frac{9}{2} + C\,.\end{aligned}$$

We note that even though the right-hand side of the two indefinite integrals look different, they are in fact the same since C could be any constant, and $\dfrac{9}{2} + C$ is also any constant.

Example 4.40. Find $\displaystyle\int \frac{2zdz}{\sqrt[3]{z^2+1}}$.

Method 1: Let $x = z^2 + 1$. Then $dx = 2zdz$; thus

$$\int \frac{2zdz}{\sqrt[3]{z^2+1}} = \int \frac{dx}{\sqrt[3]{x}} = \int x^{-\frac{1}{3}}\,dx = \frac{3}{2}x^{\frac{2}{3}} + C = \frac{3}{2}(z^2+1)^{\frac{2}{3}} + C\,.$$

Method 2: Let $y = \sqrt[3]{z^2+1}$. Then $y^3 = z^2 + 1$; thus $3y^2dy = 2zdz$. Therefore,

$$\int \frac{2zdz}{\sqrt[3]{z^2+1}} = \int \frac{3y^2dy}{y} = \int 3y\,dy = \frac{3}{2}y^2 + C = \frac{3}{2}(z^2+1)^{\frac{2}{3}} + C\,.$$

Example 4.41. Find $\displaystyle\int \frac{18\tan^2 x\sec^2 x}{(2+\tan^3 x)^2}\,dx$.

Let $u = 2 + \tan^3 x$. Then $du = 3\tan^2 x\sec^x\,dx$; thus

$$\int \frac{18\tan^2 x\sec^2 x}{(2+\tan^3 x)^2}\,dx = \int \frac{6du}{u^2} = 6\int u^{-2}\,du = -6u^{-1} + C = -\frac{6}{2+\tan^3 x} + C\,.$$

Sometimes an definite integral can be evaluated even though the anti-derivative of the integrand cannot be found. In such kind of cases, we have to look for special structures so that we can simplify the integrals. There is no general rule for this, and we have to do this case by case.

Example 4.42. Find $\displaystyle\int_0^{\pi} \frac{2x\sin x}{3+\cos(2x)}\,dx$.

Let the integral be I. By the substitution $u = \pi - x$, we find that

$$\begin{aligned}
\mathrm{I} &= \int_{\pi}^{0} \frac{2(\pi-u)\sin(\pi-u)}{3+\cos(2(\pi-u))}(-1)\,du = \int_0^{\pi} \frac{2(\pi-u)\sin u}{3+\cos 2u}\,du \\
&= \int_0^{\pi} \frac{2\pi\sin u}{3+\cos 2u}\,du - \int_0^{\pi} \frac{2u\sin u}{3+\cos 2u}\,du = 2\pi\int_0^{\pi} \frac{\sin u}{3+\cos 2u}\,du - \mathrm{I}\,;
\end{aligned}$$

thus

$$\begin{aligned}
\mathrm{I} &= \pi\int_0^{\pi} \frac{\sin u}{3+\cos 2u}\,du = -\pi\int_0^{\pi} \frac{d(\cos u)}{3+2\cos^2 u-1} = -\frac{\pi}{2}\int_1^{-1} \frac{dv}{v^2+1} \\
&= \frac{\pi}{2}\int_{-1}^{1} \frac{dv}{v^2+1} = \frac{\pi}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} \frac{\sec^2 y}{\tan^2 y+1}\,dy = \frac{\pi}{2}\int_{-\frac{\pi}{4}}^{\frac{\pi}{4}} dy = \frac{\pi^2}{4}\,.
\end{aligned}$$

Chapter 5

Logarithmic, Exponential, and other Transcendental Functions

5.1 Inverse Functions

Definition 5.1

A function g is the inverse function of the function f if

$$f(g(x)) = x \quad \text{for all } x \text{ in the domain of } g \tag{5.1.1}$$

and

$$g(f(x)) = x \quad \text{for all } x \text{ in the domain of } f. \tag{5.1.2}$$

The inverse function of f is usually denoted by f^{-1}.

Some important observations about inverse functions:

1. If g is the inverse function of f, then f is the inverse function of g.

2. Note that (5.1.1) implies that

   (a) the domain of g is contained in the range of f,

   (b) the domain of f contains the range of g,

   (c) g is one-to-one since if $g(x_1) = g(x_2)$, then $x_1 = f(g(x_1)) = f(g(x_2)) = x_2$

   and (5.1.2) implies that

   (a) the domain of f is contained in the range of g,

(b) the domain of g contains the range of f,

(c) f is one-to-one since if $f(x_1) = f(x_2)$, then $x_1 = g(f(x_1)) = g(f(x_2)) = x_2$.

According to the statements above, the domain of f^{-1} is the range of f, and the range of f^{-1} is the domain of f.

3. A function need not have an inverse function, but when it does, the inverse function is unique: Suppose that g and h are inverse function of f, then

   (a) the domain of g is identical to the domain of h (since they are both the range of f);

   (b) for each x in the range of f,

   $$f(g(x)) = x = f(h(x))$$

   thus by the fact that f is one-to-one, $g(x) = h(x)$ for all x in the range of f.

   Therefore, g and h are identical functions.

Example 5.2. The functions

$$f(x) = 2x^3 - 1 \qquad \text{and} \qquad g(x) = \sqrt[3]{\frac{x+1}{2}}$$

are inverse functions of each other since

$$f(g(x)) = 2\left[\sqrt[3]{\frac{x+1}{2}}\right]^3 - 1 = 2\frac{x+1}{2} - 1 = x$$

and

$$g(f(x)) = \sqrt[3]{\frac{2x^3-1+1}{2}} = \sqrt[3]{x^3} = x\,.$$

Theorem 5.3

A function f has an inverse function if and only if f is one-to-one.

Proof. It suffices to show the "$\Leftarrow$" direction. Suppose that f is one-to-one. Then for each x in the range of f, there exists only a unique y in the domain of f such that $f(y) = x$. Denote the map $x \mapsto y$ by g; that is,

$$y = g(x) \quad \text{if} \quad f(y) = x \text{ and } x \in \operatorname{Range}(f)\,.$$

Then $f(g(x)) = x$ for all x in the range of f. Since the domain of g is the range of f, we find that

$$f(g(x)) = x \quad \text{for all } x \text{ in the domain of } g.$$

On the other hand, by the definition of g we must also have

$$g(f(x)) = x \qquad \text{for all } x \text{ in the domain of } f;$$

thus f has an inverse function. $\square$

Theorem 5.4

Let f be a function with inverse f^{-1}. The graph of f contains the point (a, b) if and only if the graph of f^{-1} contains the point (b, a).

Proof. Let (a, b) be on the graph of f. Then $b = f(a)$ which implies that $f^{-1}(b) = f^{-1}(f(a)) = a$. Therefore, (b, a) is on the graph of f^{-1}. $\square$

Remark 5.5. Theorem 5.4 implies that the graph of f and the graph of f^{-1} is symmetric above the straight line $y = x$.

Theorem 5.6

Let f be a function defined on an interval I and have an inverse function. Then

1. if f is continuous on I, then f^{-1} is continuous on its domain;
2. if f is strictly increasing on I, then f^{-1} is strictly increasing on the range of f;
3. if f is strictly decreasing on I, then f^{-1} is strictly decreasing on the range of f;
4. if f is differentiable on an interval containing c and $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$.

Proof. We only show 2 (and the proof of 3 is similar).

To show that f^{-1} is strictly increasing on the range of f, we need to show that

$$f^{-1}(x_1) < f^{-1}(x_2) \quad \text{if } x_1 < x_2 \text{ are in the range of } f.$$

Nevertheless, if f is increasing on I and $x_1 < x_2$ are in the range of f, there exists $y_1 = f^{-1}(x_1)$ and $y_2 = f^{-1}(x_2)$ in I such that $f(y_1) = x_1$ and $f(y_2) = x_2$. Since $x_1 < x_2$, $y_1 \ngeq y_2$; thus the trichotomy law implies that $y_1 < y_2$. $\square$

Remark 5.7. If I is not an interval, then even if $f : I \to \mathbb{R}$ is one-to-one and continuous, f^{-1} might be discontinuous. For example, let $I = \left[0, \frac{\pi}{2}\right) \cup \left(\frac{\pi}{2}, \pi\right)$ and $f(x) = \tan x$. Then clearly $f : I \to \mathbb{R}$ is one-to-one, onto and continuous; however, the inverse function is not continuous at 0: you can check this by looking at the graph of f^{-1}.

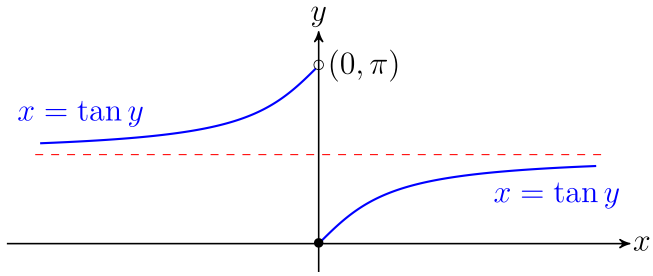


Figure 5.1: The graph of f^{-1}

From the graph of f^{-1}, we find that $\lim\limits_{x \to 0^+} f^{-1}(x) = 0$ while $\lim\limits_{x \to 0^-} f^{-1}(x) = \pi$; thus f is not continuous at 0.

Theorem 5.8: Inverse Function Differentiation

Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))} \qquad \text{for all } x \text{ with } f'(g(x)) \neq 0.$$

Proof. Suppose that f is differentiable at $g(c) \in I$ and $f'(g(c)) \neq 0$. We show that g is differentiable at c. If $k \neq 0$ is small enough, $g(c+k) - g(c) = h$. Then $c + k = f(g(c) + h)$. Moreover, $h \to 0$ as $k \to 0$ since g is continuous (by Theorem 5.6). Therefore,

$$\frac{g(c+k) - g(c)}{k} = \frac{h}{f(g(c)+h) - f(g(c))} = \frac{h}{f(g(c)+h) - f(g(c))}$$

which approaches $\dfrac{1}{f'(g(c))}$ as k approaches zero. Therefore, $g'(c) = \dfrac{1}{f'(g(c))}$. $\square$

5.2 The Function $y = \ln x$

Recall Example 4.11 that $\displaystyle\int_a^b x^q\, dx = \frac{b^{q+1} - a^{q+1}}{q+1}$ if $q \neq -1$ is a rational number and $0 < a < b$. What happened to the case $\displaystyle\int_a^b x^{-1}\, dx$? In the following, we define a new function which can be used to compute this integral.

Definition 5.9

The function $\ln : (0, \infty) \to \mathbb{R}$ is defined by

$$\ln x = \int_1^x \frac{1}{t}\, dt \qquad \forall\, x > 0\,.$$

We emphasize again that we **cannot** write $\ln x = \int_1^x \frac{1}{x}\, dx$ since the upper limit in the integral is some arbitrary but fixed number (denoted by x) and the variable of the integrand should be really arbitrary.

Remark 5.10. For historical reason, when the variable is clear we should ignore the parentheses and write $\ln x$ instead of $\ln(x)$. On the other hand, if the variable is product of several variables such as xy, for the sake of clarity we should still write $\ln(xy)$ instead of $\ln xy$.

5.2.1 Properties of $y = \ln x$

- **Differentiability**

Since the function $y = \dfrac{1}{x}$ is continuous on $(0, \infty)$, the Fundamental Theorem of Calculus implies the following

Theorem 5.11

$\dfrac{d}{dx} \ln x = \dfrac{1}{x}$ for all $x > 0$.

In particular, the function $y = \ln x$ is continuous on $(0, \infty)$.

Corollary 5.12

The function $\ln : (0, \infty) \to \mathbb{R}$ is strictly increasing on $(0, \infty)$, and the graph of $y = \ln x$ is concave downward on $(0, \infty)$.

Example 5.13. In this example we prove that

$$x - \frac{x^2}{2} \leqslant \ln(1 + x) \leqslant x \qquad \forall\, x > 0\,. \tag{5.2.1}$$

Let $f(x) = \ln(1 + x) - x + \dfrac{x^2}{2}$ and $g(x) = \ln(1 + x) - x$. Then for $x > 0$,

$$f'(x) = \frac{1}{1 + x} - 1 + x = \frac{x^2}{1 + x} > 0\,, \qquad g'(x) = \frac{1}{1 + x} - 1 = \frac{-x}{1 + x} < 0\,.$$

The two identities above shows that f is strictly increasing on $[0,\infty)$ and g is strictly decreasing on $[0,\infty)$. Therefore,

$$f(x) > f(0) = 0 \quad \text{and} \quad g(x) < g(0) = 0 \qquad \forall\, x > 0\,.$$

These inequalities lead to (5.2.1).

- **The range**

Next we show that $\lim\limits_{x\to\infty} \ln x = \infty$ and $\lim\limits_{x\to-\infty} \ln x = -\infty$. To see this, we note that

$$\begin{aligned}
\ln(2^n) &= \int_1^{2^n} \frac{1}{t}\,dt = \int_1^2 \frac{1}{t}\,dt + \int_2^4 \frac{1}{t}\,dt + \int_4^8 \frac{1}{t}\,dt + \cdots + \int_{2^{n-1}}^{2^n} \frac{1}{t}\,dt \\
&= \sum_{i=1}^n \int_{2^{i-1}}^{2^i} \frac{1}{t}\,dt \geqslant \sum_{i=1}^n \int_{2^{i-1}}^{2^i} \frac{1}{2^i}\,dt = \sum_{i=1}^n \frac{2^i - 2^{i-1}}{2^i} = \sum_{i=1}^n \frac{1}{2} = \frac{n}{2}
\end{aligned}$$

and

$$\begin{aligned}
\ln(2^{-n}) &= \int_1^{2^{-n}} \frac{1}{t}\,dt = -\int_{2^{-n}}^1 \frac{1}{t}\,dt = -\Big[\int_{2^{-n}}^{2^{-n+1}} \frac{1}{t}\,dt + \int_{2^{-n+1}}^{2^{-n+2}} \frac{1}{t}\,dt + \cdots + \int_{\frac{1}{2}}^1 \frac{1}{t}\,dt\Big] \\
&= -\sum_{i=1}^n \int_{2^{-i}}^{2^{1-i}} \frac{1}{t}\,dt \leqslant -\sum_{i=1}^n \int_{2^{-i}}^{2^{1-i}} \frac{1}{2^{1-i}}\,dt = -\sum_{i=1}^n \frac{2^{1-i} - 2^{-i}}{2^{1-i}} = -\sum_{i=1}^n \frac{1}{2} = -\frac{n}{2}\,;
\end{aligned}$$

thus we have $\lim\limits_{x\to\infty} \ln x = \infty$ and $\lim\limits_{x\to-\infty} \ln x = -\infty$. By the continuity of ln and the Intermediate Value Theorem, for each $b \in \mathbb{R}$ there exists one $a \in (0,\mathbb{R})$ such that $b = \ln a$. By the strict monotonicity $\ln : (0,\infty) \to \mathbb{R}$ is one-to-one and onto.

Remark 5.14. In particular, there exists one unique number e such that $\ln e = 1$. We note that

$$\ln 2 = \int_1^2 \frac{1}{t}\,dt = \int_1^{1.5} \frac{1}{t}\,dt + \int_{1.5}^2 \frac{1}{t}\,dt \leqslant \frac{0.5}{1} + \frac{0.5}{1.5} = \frac{5}{6} < 1$$

and

$$\begin{aligned}
\ln 3 &= \int_1^3 \frac{1}{t}\,dt = \Big(\int_1^{1.25} + \int_{1.25}^{1.5} + \int_{1.5}^{1.75} + \int_{1.75}^2 + \int_2^{2.5} + \int_{2.5}^3\Big)\frac{1}{t}\,dt \\
&\geqslant \frac{0.25}{1.25} + \frac{0.25}{1.5} + \frac{0.25}{1.75} + \frac{0.25}{2} + \frac{0.5}{2.5} + \frac{0.5}{3} \\
&= \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} + \frac{1}{5} + \frac{1}{6} = \frac{841}{840} > 1\,.
\end{aligned}$$

Therefore, $2 < e < 3$. In fact, $e \approx 2.718281828459$.

Example 5.15. In this example we show that there is no slant/horizontal asymptote of the graph of $y = \ln x$. Recall that if the graph of $y = \ln x$ has a slant/horizontal asymptote $y = mx + k$, then $m = \lim\limits_{x\to\infty} \dfrac{\ln x}{x}$ and $k = \lim\limits_{x\to\infty} (\ln x - mx)$. We first show that $\lim\limits_{x\to\infty} \dfrac{\ln x}{x} = 0$.

Let $\varepsilon > 0$. Choose $M = \max\left\{\dfrac{\varepsilon}{2}, 1\right\}$. Then if $x > M$, for all $1 < c < x$ we have

$$0 < \frac{\ln x}{x} = \frac{1}{x}\int_1^x \frac{1}{t}\,dt = \frac{1}{x}\Big[\int_1^c \frac{1}{t}\,dt + \int_c^x \frac{1}{t}\,dt\Big] \leqslant \frac{c-1}{x} + \frac{1}{x}\int_c^x \frac{1}{t}\,dt\,.$$

By the mean value theorem for integrals (Theorem 4.26), there exists $c \leqslant d \leqslant x$ such that $\displaystyle\int_c^x \frac{1}{t}\,dt = \frac{x-c}{d}$; thus if $x > M$ and $1 < c < x$,

$$0 < \frac{\ln x}{x} = \frac{1}{x}\int_1^x \frac{1}{t}\,dt \leqslant \frac{c-1}{x} + \frac{x-c}{dx} \leqslant \frac{c-1}{M} + \frac{1}{M} \leqslant \frac{\varepsilon c}{2} < \varepsilon\,,$$

where the last inequality is concluded by choosing $1 < c < x$ and $c < 2$. Therefore, for every $\varepsilon > 0$ there exists $M > 0$ such that

$$\Big|\frac{\ln x}{x} - 0\Big| < \varepsilon \qquad \text{whenever} \quad x > M\,.$$

This is exactly the definition of $\lim\limits_{x\to\infty} \dfrac{\ln x}{x} = 0$. However, since the range of ln is $\mathbb{R}$, $\lim\limits_{x\to\infty} \ln x = \infty$ which implies that

$$\lim_{x\to 0}(\ln x - 0 \cdot x) \text{ D.N.E.}$$

Therefore, there is no slant/horizontal asymptote of the graph of $y = \ln x$.

- **Logarithmic Laws**

The most important property of the function $y = \ln x$ is the relation among $\ln a$, $\ln b$ and $\ln(ab)$. By the property of integration,

$$\ln(ab) = \int_1^{ab} \frac{1}{t}\,dt = \int_1^a \frac{1}{t}\,dt + \int_a^{ab} \frac{1}{t}\,dt = \ln a + \int_a^{ab} \frac{1}{t}\,dt\,.$$

By the substitution $t = au$, $dt = adu$; thus

$$\int_a^{ab} \frac{1}{t}\,dt = \int_1^b \frac{1}{au} a du = \int_1^b \frac{1}{u}\,du = \ln b\,.$$

Therefore, we obtain the identity:

$$\ln(ab) = \ln a + \ln b \qquad \forall\, a, b > 0\,. \tag{5.2.2}$$

Having established (5.2.2), we can show that the function ln is a logarithmic function for the following reason. First, we observe that for all $a > 0$ and $n \in \mathbb{N}$,

$$\ln(a^n) = \ln(a^{n-1}a) = \ln(a^{n-1}) + \ln a = \ln(a^{n-2}a) + \ln a = \ln(a^{n-2}) + 2\ln a = \cdots = n\ln a\,.$$

Moreover, by the definition of ln, $0 = \ln(1) = \ln(a^0) = 0\ln a$; thus

$$\ln(a^n) = n\ln a \qquad \forall\, a > 0\,, n \in \mathbb{N} \cup \{0\}\,.$$

Next, by the law of exponents, for $a > 0$ and $n \in \mathbb{N}$ we have

$$0 = \ln(a^0) = \ln(a^n \cdot a^{-n}) = \ln(a^n) + \ln(a^{-n}) = n\ln a + \ln(a^{-n})\,.$$

Therefore, for all $n \in \mathbb{N}$, we also have $\ln(a^{-n}) = -n\ln a$; hence

$$\ln(a^n) = n\ln a \qquad \forall\, a > 0\,, n \in \mathbb{Z}\,.$$

The identity above also implies that if $k, n \in \mathbb{Z}$ and $n \neq 0$,

$$n\ln(a^{\frac{k}{n}}) = \ln((a^{\frac{k}{n}})^n) = \ln(a^k) = k\ln a\,,$$

and this shows that

$$\ln(a^{\frac{k}{n}}) = \frac{k}{n}\ln a \qquad \forall\, a > 0\,, n, k \in \mathbb{Z}\,, n \neq 0\,.$$

As a consequence,

$$\ln(a^r) = r\ln a \qquad \forall\, a > 0\,, r \in \mathbb{Q}\,.$$

Finally, we find that $\ln(e^r) = r\ln e = r$, so $\ln x$ is indeed the logarithm of x to the base e. In other words, we obtain that

$$\log_e x = \ln x = \int_1^x \frac{1}{t}\,dt \qquad \forall\, x > 0\,. \tag{5.2.3}$$

Theorem 5.16: Logarithmic properties of $y = \ln x$

Let a, b be positive numbers and r be a rational number. Then

1. $\ln 1 = 0$;
2. $\ln(ab) = \ln a + \ln b$;
3. $\ln(a^r) = r\ln a$;
4. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$.

Remark 5.17. Since the function $y = \ln x$ has the logarithmic property, it is called the **natural logarithmic** function.

Example 5.18. Let $f(x) = \dfrac{(x^2+3)^2}{x\sqrt[3]{x^2+1}}$. Since $\ln f(x) = 2\ln(x^2+3) - \ln x - \dfrac{1}{3}\ln(x^2+1)$ for $x > 0$, by the chain rule we find that

$$\frac{f'(x)}{f(x)} = \frac{d}{dx}\ln f(x) = \frac{4x}{x^2+3} - \frac{1}{x} - \frac{2x}{3(x^2+1)};$$

thus

$$f'(x) = \frac{(x^2+3)^2}{x\sqrt[3]{x^2+1}}\Big[\frac{d}{dx}\ln f(x) = \frac{4x}{x^2+3} - \frac{1}{x} - \frac{2x}{3(x^2+1)}\Big].$$

Theorem 5.19

If f is a differentiable function on an interval I, then $\ln|f|$ is differentiable at those point $x \in I$ satisfying $f(x) \neq 0$. Moreover,

$$\frac{d}{dx}\ln\big|f(x)\big| = \frac{f'(x)}{f(x)} \qquad \text{for all } x \in I \text{ with } f(x) \neq 0\,.$$

Proof. Note that the function $y = |x|$ is differentiable at non-zero points, and

$$\frac{d}{dx}|x| = \frac{d}{dx}(x^2)^{\frac{1}{2}} = \frac{1}{2}(x^2)^{-\frac{1}{2}} \cdot 2x = \frac{x}{|x|} \qquad \forall\, x \neq 0\,.$$

If $f(c) \neq 0$, by the fact that the natural logarithmic function ln is differentiable at $|f(c)|$, the absolute function $|\cdot|$ is differentiable at $f(c)$ and f is differentiable at c, the chain rule implies that $y = \ln\big|f(x)\big|$ is differentiable at c and

$$\frac{d}{dx}\Big|_{x=c}\ln\big|f(x)\big| = \frac{1}{\big|f(c)\big|}\frac{f(c)}{\big|f(c)\big|}f'(c) = \frac{f'(c)}{f(c)}\,. \qquad \square$$

Example 5.20. $\dfrac{d}{dx}\ln|\cos x| = \dfrac{-\sin x}{\cos x} = -\tan x$ for all x with $\cos x \neq 0$.

Example 5.21. Compute the derivative of $f(x) = \dfrac{(x^2+3)^2}{x\sqrt[3]{x^2+1}}$ for $x > 0$.

Let $h(x) = \ln f(x)$. Then

$$\begin{aligned}\frac{f'(x)}{f(x)} = h'(x) &= \frac{d}{dx}\Big[2\ln(x^2+3) - \ln x - \frac{1}{3}\ln(x^2+1)\Big] \\ &= 2\frac{d}{dx}\ln(x^2+3) - \frac{d}{dx}\ln x - \frac{1}{3}\frac{d}{dx}\ln(x^2+1) \\ &= \frac{4x}{x^2+3} - \frac{1}{x} - \frac{2x}{3(x^2+1)};\end{aligned}$$

thus

$$f'(x) = \frac{(x^2+3)^2}{x\sqrt[3]{x^2+1}}\Big[\frac{4x}{x^2+3} - \frac{1}{x} - \frac{2x}{3(x^2+1)}\Big].$$

5.3 Integrations Related to $y = \ln x$

Theorem 5.19 implies the following

Theorem 5.22

1. $\displaystyle\int \frac{1}{x}\,dx = \ln|x| + C$; 2. $\displaystyle\int \frac{f'(x)}{f(x)}\,dx = \ln\big|f(x)\big| + C$.

Example 5.23. Compute $\displaystyle\int \frac{x}{x^2+1}\,dx$. From observation, the numerator is a half of the derivative of the denominator, so

$$\int \frac{x}{x^2+1}\,dx = \frac{1}{2}\int \frac{2x}{x^2+1}\,dx = \frac{1}{2}\ln(x^2+1) + C\,.$$

Example 5.24. Compute $\displaystyle\int \frac{1}{x\ln x}\,dx$. Let $u = \ln x$. Then $du = \frac{1}{x}\,dx$; thus

$$\int \frac{1}{x\ln x}\,dx = \int \frac{1}{u}\,du = \ln|u| + C = \ln|\ln x| + C\,.$$

Theorem 5.25

1. $\displaystyle\int \sin x\,dx = -\cos x + C$; 2. $\displaystyle\int \cos x\,dx = \sin x + C$;

3. $\displaystyle\int \tan x\,dx = -\ln|\cos x| + C = \ln|\sec x| + C$;

4. $\displaystyle\int \sec x\,dx = \ln|\sec x + \tan x| + C$.

Proof. We only prove 4. Let $t = \tan\frac{x}{2}$. Then $\sin x = \frac{2t}{1+t^2}$, $\cos x = \frac{1-t^2}{1+t^2}$ and $dx = \frac{2dt}{1+t^2}$; thus

$$\begin{aligned}\int \sec x\,dx &= \int \frac{1+t^2}{1-t^2}\frac{2}{1+t^2}\,dt = \int \frac{2}{1-t^2}\,dt = \int \frac{-2}{(t-1)(t+1)}\,dt \\ &= \int \Big[\frac{1}{t+1} - \frac{1}{t-1}\Big]\,dt = \ln|t+1| - \ln|t-1| + C = \ln\Big|\frac{t+1}{t-1}\Big| + C\,.\end{aligned}$$

The conclusion then follows from the identity

$$\frac{t+1}{t-1} = \frac{\sin\frac{x}{2}+\cos\frac{x}{2}}{\sin\frac{x}{2}-\cos\frac{x}{2}} = \frac{\big(\sin\frac{x}{2}+\cos\frac{x}{2}\big)^2}{\sin^2\frac{x}{2}-\cos^2\frac{x}{2}} = \frac{1+2\sin\frac{x}{2}\cos\frac{x}{2}}{-\cos x}$$
$$= -\frac{1+\sin x}{\cos x} = -(\sec x + \tan x)\,. \qquad \square$$

Finally we compute $\displaystyle\int_1^a \ln x\,dx$ for $a>0$. Suppose first that $a>1$. Following the idea of Example 4.5, we let $r = a^{\frac{1}{n}}$ and $x_i = r^i$, as well as a partition $\mathcal{P} = \{1 = x_0 < x_1 < \cdots < x_n = a\}$ of $[1,a]$. Then the Riemann sum of f for the partition $\mathcal{P}$ given by the right end-point rule, which happens to be the upper sum of f for the partition $\mathcal{P}$, is

$$S(\mathcal{P}) = \sum_{i=1}^n \ln(x_i)(x_i - x_{i-1}) = \sum_{i=1}^n \ln(r^i)(r^i - r^{i-1}) = (r-1)\ln r \sum_{i=1}^n i r^{i-1}\,.$$

Note that $ir^{i-1} = \dfrac{d}{dr} r^i$; thus

$$\sum_{i=1}^n i r^{i-1} = \sum_{i=1}^n \frac{d}{dr} r^i = \frac{d}{dr}\sum_{i=1}^n r^i = \frac{d}{dr}\frac{r^{n+1}-r}{r-1} = \frac{\big[(n+1)r^n - 1\big](r-1) - r^{n+1} + r}{(r-1)^2}$$
$$= \frac{nr^{n+1} - (n+1)r^n + 1}{(r-1)^2} = \frac{nar - (n+1)a + 1}{(r-1)^2}\,.$$

By the fact that $n = \dfrac{\ln a}{\ln r}$,

$$S(\mathcal{P}) = \frac{ra\ln a - a\ln a - a\ln r + \ln r}{r-1}\,.$$

Since $\|\mathcal{P}\| \to 0$ is equivalent to that $r \to 1$,

$$\lim_{\|\mathcal{P}\|\to 0} S(\mathcal{P}) = \lim_{r\to 1} \frac{ra\ln a - a\ln a - a\ln r + \ln r}{r-1} = \frac{d}{dr}\Big|_{r=1}\big(ra\ln a - a\ln a - a\ln r + \ln r\big)$$
$$= a\ln a - a + 1\,.$$

If $0 < a < 1$, by Remark 4.16 it suffices to show that $a^{\frac{1}{n}} \to 1$ as n approaches infinity. Nevertheless, $a^{\frac{1}{n}} = 1/(1/a)^{\frac{1}{n}}$ and the denominator approaches 1 as n approaches infinity; thus $\lim\limits_{n\to\infty} a^{\frac{1}{n}} = 1$ even if $0 < a < 1$.

Theorem 5.26

1. $\displaystyle\int_1^a \ln x\,dx = a\ln a - a + 1$ for all $a > 0$;
2. $\displaystyle\int \ln x\,dx = x\ln x - x + C$.

Example 5.27. Find the limit $\lim\limits_{n\to\infty}\Big(\dfrac{n!}{n^n}\Big)^{\frac{1}{n}}$.

Consider the sum $\sum\limits_{k=1}^{n}\dfrac{1}{n}\ln\dfrac{k}{n}$. This sum looks like a Riemann sum of the "integral" $\displaystyle\int_0^1 \ln x\,dx$; however, since $\ln x$ blows up at $x=0$, $\ln x$ is not Riemann integrable on $[0,1]$. In other words, the sum is not a Riemann sum for a particular integral.

On the other hand, by the monotonicity of the function $y=\ln x$, we find that

$$\sum_{k=1}^{n}\frac{1}{n}\ln\frac{k}{n}=\sum_{k=1}^{n-1}\frac{1}{n}\ln\frac{k}{n}\leqslant\int_{\frac{1}{n}}^{1}\ln x\,dx\leqslant\sum_{k=2}^{n}\frac{1}{n}\ln\frac{k}{n}=-\frac{1}{n}\ln\frac{1}{n}+\sum_{k=1}^{n}\frac{1}{n}\ln\frac{k}{n};$$

thus by Theorem 5.26,

$$\frac{1}{n}-1\leqslant\sum_{k=1}^{n}\frac{1}{n}\ln\frac{k}{n}\leqslant-\frac{1}{n}\ln\frac{1}{n}+\frac{1}{n}-1\,.$$

Therefore, by the fact that $\lim\limits_{n\to\infty}\dfrac{\ln n}{n}=0$, we conclude from the Squeeze Theorem that

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n}\ln\frac{k}{n}=-1\,.$$

Finally, note that

$$\sum_{k=1}^{n}\frac{1}{n}\ln\frac{k}{n}=\frac{1}{n}\sum_{k=1}^{n}\ln\frac{k}{n}=\frac{1}{n}\ln\frac{n!}{n^n}=\ln\Big(\frac{n!}{n^n}\Big)^{\frac{1}{n}};$$

thus the continuity and strict monotonicity of $y=\ln x$ implies that

$$\lim_{n\to\infty}\Big(\frac{n!}{n^n}\Big)^{\frac{1}{n}}=\frac{1}{e}\,.$$

5.4 Exponential Functions

In the previous section we have shown that the natural logarithmic function $\ln:(0,\infty)\to\mathbb{R}$ is one-to-one and onto. Therefore, for each $a\in\mathbb{R}$ there exists a unique $b\in(0,\infty)$ satisfying $a=\ln b$. The map $a\mapsto b$ is called the natural exponential function. To be more precise, we have the following

Definition 5.28

The natural exponential function $\exp:\mathbb{R}\to(0,\infty)$ is a function defined by

$$\exp(x)=y\qquad\text{if and only if}\qquad x=\ln y\,.$$

By the definition of the natural exponential function, we have

$$\exp(\ln x) = x \quad \forall\, x \in (0,\infty) \qquad \text{and} \qquad \ln(\exp(x)) = x \quad \forall\, x \in \mathbb{R}\,. \tag{5.4.1}$$

Therefore, exp and ln are inverse functions to each other; thus $\exp : \mathbb{R} \to (0,\infty)$ is one-to-one, onto, and strictly increasing. Note that by the definition, $\exp(0) = 1$.

Let $a > 0$ be a real number. If $r \in \mathbb{Q}$, a^r is a well-defined positive number and the logarithmic laws implies that

$$\ln a^r = r \ln a\,.$$

By the definition of the natural exponential function, $a^r = \exp(r \ln a)$ for all $r \in \mathbb{Q}$. Since $\exp : \mathbb{R} \to (0,\infty)$ is continuous, for a real number x, we shall defined a^x as $\exp(x \ln a)$ and this induces the following

Definition 5.29

Let $a > 0$ be a real number. For each $x \in \mathbb{R}$, the exponential function to the base a, denote by $y = a^x$, is defined by $a^x \equiv \exp(x \ln a)$. In other words,

$$a^x = \exp(x \ln a) \qquad \forall\, x \in \mathbb{R}\,.$$

Remark 5.30. For each $x \in \mathbb{R}$, the number 1^x is 1 since $1^x = \exp(x \ln 1) = \exp(0) = 1$.

Remark 5.31. The function $y = e^x$ is identical to the function $y = \exp(x)$ since

$$e^x = \exp(x \ln e) = \exp(x) \qquad \forall\, x \in \mathbb{R}\,.$$

Therefore, we often write $\exp(x)$ as e^x as well (even though e^x, when x is a irrational number, has to be defined through the natural exponential function), and write $a^x = e^{x \ln a}$. Moreover, by the definition of the natural exponential function,

$$\ln(a^x) = \ln(\exp(x \ln a)) = x \ln a \qquad \forall\, a > 0 \text{ and } x \in \mathbb{R}\,. \tag{5.4.2}$$

5.4.1 Properties of Exponential Functions

- **The range and the strict monotonicity of the exponential functions**

Note that Theorem 5.6 implies that $\exp : \mathbb{R} \to (0,\infty)$ is strictly increasing. Suppose that $a > 1$. Then $\ln a > 0$ which further implies that

$$a^{x_1} = \exp(x_1 \ln a) < \exp(x_2 \ln a) = a^{x_2} \qquad \forall\, x_1 < x_2\,.$$

Similarly, if $0 < a < 1$, the exponential function to the base a is a strictly decreasing function.

Moreover, since $\exp : \mathbb{R} \to (0, \infty)$ is onto, we must have that for $0 < a \neq 1$, the range of the exponential function to the base a is also $\mathbb{R}$. Therefore, for $0 < a \neq 1$, the exponential function $a^{\cdot} : \mathbb{R} \to (0, \infty)$ is one-to-one and onto.

- **The law of exponentials**

(a) If $a > 0$, then $a^{x+y} = a^x a^y$ for all $x, y \in \mathbb{R}$: First we show the case when $a = e$. Let $\exp(x) = c$ and $\exp(y) = d$ or equivalently, $x = \ln c$ and $y = \ln d$. Then

$$e^{x+y} = \exp(x+y) = \exp(\ln c + \ln d) = \exp(\ln(cd)) = cd = e^x e^y .$$

For general $a > 0$, by the definition of exponential functions, for $x, y \in \mathbb{R}$,

$$a^{x+y} = \exp((x+y)\ln a) = e^{x\ln a + y\ln a} = e^{x\ln a}e^{y\ln a} = \exp(x\ln a)\exp(y\ln a) = a^x a^y .$$

(b) If $a > 0$, then $a^{x-y} = \dfrac{a^x}{a^y}$ for all $x, y \in \mathbb{R}$: Using (a), we obtain that

$$a^{x-y}a^y = a^{x-y+y} = a^x \qquad \forall\, x, y \in \mathbb{R}\,;$$

thus $a^{x-y} = \dfrac{a^x}{a^y}$ for all $x, y \in \mathbb{R}$.

(c) If $a, b > 0$, then $(ab)^x = a^x b^x$ for all $x \in \mathbb{R}$: By the definition of the exponential functions,

$$(ab)^x = e^{x\ln(ab)} = e^{x(\ln a + \ln b)} = e^{x\ln a + x\ln b} = e^{x\ln a}e^{x\ln b} = a^x b^x .$$

(d) If $a, b > 0$, then $\Big(\dfrac{a}{b}\Big)^x = \dfrac{a^x}{b^x}$ for all $x \in \mathbb{R}$: Using (b), we obtain that

$$\Big(\frac{a}{b}\Big)^x = e^{x\ln\frac{a}{b}} = e^{x\ln(ab^{-1})} = e^{x(\ln a - \ln b)} = \frac{e^{x\ln a}}{e^{x\ln b}} = \frac{a^x}{b^x} .$$

(e) If $a > 0$, then $(a^x)^y = a^{xy}$ for all $x, y \in \mathbb{R}$: Using (5.4.2),

$$(a^x)^y = e^{y\ln a^x} = e^{xy\ln a} = a^{xy} .$$

- **The differentiation of the exponential functions**

Theorem 5.32

$\dfrac{d}{dx}e^x = e^x$ for all $x \in \mathbb{R}$.

Proof. Define $f : (0, \infty) \to \mathbb{R}$ and $g : \mathbb{R} \to (0, \infty)$ by $f(x) = \ln x$ and $g(x) = \exp(x) = e^x$. Then f and g are inverse functions to each other, and the Inverse Function Differentiation implies that

$$g'(x) = \frac{1}{f'(g(x))} \qquad \forall\, x \in \mathbb{R} \text{ with } f'(g(x)) \neq 0\,.$$

Since $f'(x) = \dfrac{1}{x}$, $f'(g(x)) = \dfrac{1}{g(x)} = \exp(-x) \neq 0$ for all $x \in \mathbb{R}$; thus

$$g'(x) = g(x) \qquad \forall\, x \in \mathbb{R}\,.$$

$\square$

Corollary 5.33

1. $\displaystyle\int_0^a e^x\,dx = e^a - 1$ for all $a > 0$;
2. $\displaystyle\int e^x\,dx = e^x + C$.

The following corollary is a direct consequence of Theorem 5.32 and the chain rule.

Corollary 5.34

Let f be a differentiable function defined on an interval I. Then

$$\frac{d}{dx}e^{f(x)} = e^x f'(x) \qquad \forall\, x \in I\,.$$

Corollary 5.35

1. For $a > 0$, $\dfrac{d}{dx}a^x = a^x \ln a$ for all $x \in \mathbb{R}$ $\Big($so $\displaystyle\int a^x\,dx = \frac{a^x}{\ln a} + C\Big)$.
2. Let r be a real number. Then $\dfrac{d}{dx}x^r = rx^{r-1}$ for all $x > 0$.
3. Let f, g be differentiable functions defined on an interval I. Then

$$\frac{d}{dx}|f(x)|^{g(x)} = |f(x)|^{g(x)}\Big[g'(x)\ln|f(x)| + \frac{f'(x)}{f(x)}g(x)\Big] \qquad \forall\, x \in I \text{ with } f(x) \neq 0\,.$$

Proof. The corollary holds because $a^x = e^{x \ln a}$, $x^r = e^{r \ln x}$, and $|f(x)|^{g(x)} = e^{g(x)\ln|f(x)|}$. $\square$

Example 5.36. $\dfrac{d}{dx}e^{-\frac{3}{x}} = e^{-\frac{3}{x}}\dfrac{d}{dx}\Big(-\dfrac{3}{x}\Big) = \dfrac{3e^{-3/x}}{x^2}$ for all $x \neq 0$.

Example 5.37. Let $f : (0, \infty) \to \mathbb{R}$ be defined by $f(x) = x^x$. Then

$$f'(x) = \frac{d}{dx}e^{x\ln x} = e^{x\ln x}\frac{d}{dx}(x\ln x) = x^x(\ln x + 1)\,.$$

Example 5.38. Find the indefinite integral $\displaystyle\int 5xe^{-x^2}\,dx$.

Let $u = -x^2$. Then $du = -2xdx$; thus

$$\int 5xe^{-x^2}\,dx = -\frac{5}{2}\int e^{-x^2}(-2x)\,dx = -\frac{5}{2}\int e^u\,du = -\frac{5}{2}e^u + C = -\frac{5}{2}e^{-x^2} + C\,.$$

Example 5.39. Compute the definite integral $\displaystyle\int_{-1}^{0} e^x\cos(e^x)\,dx$.

Let $u = e^x$. Then $du = e^x\,dx$; thus

$$\int_{-1}^{0} e^x\cos(e^x)\,dx = \int_{e^{-1}}^{1}\cos u\,du = \sin u\Big|_{u=e^{-1}}^{u=1} = \sin 1 - \sin(e^{-1})\,.$$

5.4.2 The number e

By the mean value theorem for integrals, for each $x > 0$ there exists $c \in [1, 1+x]$ such that

$$\frac{\ln(1+x)}{x} = \frac{1}{x}\int_1^{1+x}\frac{1}{t}\,dt = \frac{1}{c}$$

which implies that

$$(1+x)^{\frac{1}{x}} = \exp\big(\ln(1+x)^{\frac{1}{x}}\big) = \exp\Big(\frac{\ln(1+x)}{x}\Big) = \exp\big(\frac{1}{c}\big)\,.$$

By the fact that the natural exponential function is continuous, we find that

$$\lim_{x\to 0^+}(1+x)^{\frac{1}{x}} = \lim_{x\to 0^+}\exp\big(\frac{1}{c}\big) = \lim_{c\to 1^+}\exp\big(\frac{1}{c}\big) = e\,.$$

Note that the limit above also shows that

$$e = \lim_{x\to\infty}\big(1 + \frac{1}{x}\big)^x\,. \tag{5.4.3}$$

Example 5.40. Let $f(x) = (1+x)^{\frac{1}{x}} = e^{\frac{\ln(1+x)}{x}}$. Then

$$f'(x) = (1+x)^{\frac{1}{x}}\cdot\frac{\dfrac{x}{1+x} - \ln(1+x)}{x^2} = \frac{(1+x)^{\frac{1}{x}}}{x^2}\Big(1 - \frac{1}{1+x} - \ln(1+x)\Big)\,.$$

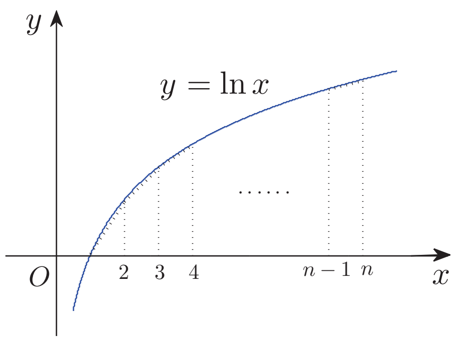
Let $g(x) = 1 - \dfrac{1}{1+x} - \ln(1+x)$. Then

$$g'(x) = \frac{1}{(1+x)^2} - \frac{1}{1+x} = \frac{-x}{(1+x)^2} < 0 \qquad \text{if } x > 0\,.$$

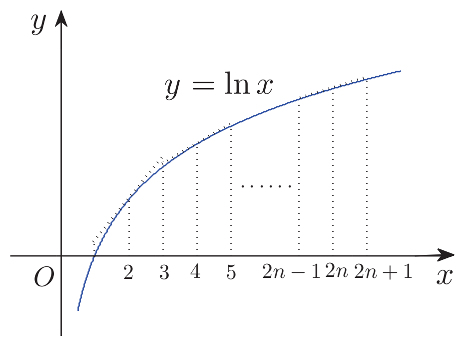
Therefore, $g(x) < g(0) = 0$ if $x > 0$; thus $f'(x) < 0$ for $x > 0$. This implies that f is strictly decreasing on $(0, \infty)$. This fact then implies that the function $h(x) = \big(1 + \dfrac{1}{x}\big)^x$ is strictly increasing on $(0, \infty)$.

Example 5.41. From Example 5.27 we find that for large n we have $\Big(\dfrac{n!}{n^n}\Big)^{\frac{1}{n}} \approx \dfrac{1}{e}$ which seems to imply that $n! \approx n^n e^{-n}$. This is in fact not true since the n-root of any constant, or even n, converges to 1. In this example, we try to determine how $n!$ behaves as $n \to \infty$.

Recall that the graph of $y = \ln x$ is concave downward. Therefore, we have the two figures below



(a) Under-estimate of $\displaystyle\int_1^n \ln x dx$



(b) Over-estimate of $\displaystyle\int_1^{2n+1} \ln x dx$

and find that

$$\int_1^n \ln x\,dx \geqslant \sum_{k=2}^{n} \frac{\ln k + \ln(k-1)}{2} = \frac{1}{2}\sum_{k=2}^{n} \ln k + \frac{1}{2}\sum_{k=1}^{n-1} \ln k = \ln(n!) - \frac{1}{2}\ln n$$

and

$$\int_1^{2n+1} \ln x\,dx \leqslant \sum_{k=1}^{n} 2\ln(2k) = 2n\ln 2 + 2\sum_{k=1}^{n} \ln k = 2n\ln 2 + 2\ln(n!)\,.$$

Theorem 5.26 then shows that

$$\ln(n!) - \frac{1}{2}\ln n \leqslant n\ln n - n + 1 \qquad \text{and} \qquad \big(n + \frac{1}{2}\big)\ln\big(n + \frac{1}{2}\big) + \frac{1}{2}\ln 2 - n \leqslant \ln(n!)\,.$$

As a consequence, we conclude that

$$\sqrt{2}\big(1 + \frac{1}{2n}\big)^{n+0.5} \leqslant \frac{n!}{n^{n+0.5}e^{-n}} \leqslant e \qquad \forall\, n \in \mathbb{N}\,. \tag{5.4.4}$$

Note that the function $f(x) = \left(1 + \frac{1}{2x}\right)^{x+0.5}$ is decreasing on $(0, \infty)$ since (5.2.1) shows that

$$f'(x) = f(x)\frac{d}{dx}\Big[\Big(x + \frac{1}{2}\Big)\ln\Big(1 + \frac{1}{2x}\Big)\Big] = f(x)\Big[\ln\Big(1 + \frac{1}{2x}\Big) - \frac{1}{2x}\Big] \leqslant 0 \quad \text{for all } x > 0\,;$$

thus (5.4.3) and (5.4.4) imply that

$$\sqrt{2e} \leqslant \frac{n!}{n^{n+0.5}e^{-n}} \leqslant e \qquad \forall\, n \in \mathbb{N}\,.$$

5.5 Logarithmic Functions to Bases Other than e

Definition 5.42

Let $0 < a \neq 1$ be a real number. The logarithmic function to the base a, denoted by $\log_a$, is the inverse function of the exponential function to the base a. In other words,

$$y = \log_a x \qquad \text{if and only if} \qquad a^y = x\,.$$

Theorem 5.43

Let $0 < a \neq 1$. Then $\log_a x = \dfrac{\ln x}{\ln a}$ for all $x > 0$.

Proof. Let $y = \log_a x$. Then $a^y = x$; thus (5.4.2) implies that

$$y \ln a = \ln(a^y) = \ln x$$

which shows $y = \dfrac{\ln x}{\ln a}$. □

5.5.1 Properties of logarithmic functions

- **Logarithmic laws**

The following theorem is a direct consequence of Theorem 5.16 and 5.43.

Theorem 5.44: Logarithmic properties of $y = \log_a x$

Let a, b, c be positive numbers, $a \neq 1$, and r is rational. Then

1. $\log_a 1 = 0$;
2. $\log_a(bc) = \log_a b + \log_b c$;
3. $\log_a(a^x) = x$ for all $x \in \mathbb{R}$;
4. $a^{\log_a x} = x$ for all $x > 0$;
5. $\log_a \left(\dfrac{b}{c}\right) = \log_a b - \log_a c$.

• **The change of base formula**

We have the following identity

$$\log_a c = \frac{\log_b c}{\log_b a} \qquad \forall\, a, b, c > 0, a, b \neq 1\,.$$

In fact, if $d = \log_a c$, then $c = a^d$; thus $\log_b c = d \log_b a$ which implies the identity above.

• **The differentiation of** $y = \log_a x$

By Theorem 5.43, we find that

$$\frac{d}{dx}\log_a x = \frac{1}{x\ln a} \qquad \forall\, x > 0\,.$$

Similar to Theorem 5.19, if f is differentiable on an interval I, we also have

$$\frac{d}{dx}\log_a \big|f(x)\big| = \frac{f'(x)}{f(x)\ln a} \qquad \text{for all } x \in I \text{ with } f(x) \neq 0\,.$$

5.6 Indeterminate Forms and L'Hôspital's Rule

Theorem 5.45: Cauchy Mean Value Theorem

Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $g'(x) \neq 0$ for all $x \in (a, b)$, then there exists $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}\,.$$

Proof. Let $h : [a, b] \to \mathbb{R}$ be defined by

$$h(x) = \big(f(x) - f(a)\big)\big(g(b) - g(a)\big) - \big(f(b) - f(a)\big)\big(g(x) - g(a)\big)\,.$$

Then $h(a) = h(b) = 0$, and h is differentiable on (a, b). Then Rolle's Theorem implies that there exists $c \in (a, b)$ such that $h'(c) = 0$; thus for some $c \in (a, b)$,

$$f'(c)\big(g(b) - g(a)\big) - \big(f(b) - f(a)\big)g'(c) = 0\,.$$

Since $g'(x) \neq 0$ for all $x \in (a, b)$, the Mean Value Theorem implies that $g(b) \neq g(a)$. Therefore, the equality above implies that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

for some $c \in (a, b)$. $\square$

Theorem 5.46: L'Hôspital's Rule

Let f, g be differentiable on (a,b), and $\dfrac{f(x)}{g(x)}$ and $\dfrac{f'(x)}{g'(x)}$ be defined on (a,b). If $\lim\limits_{x\to a^+}\dfrac{f'(x)}{g'(x)}$ exists, and one of the following conditions holds:

1. $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^+} g(x) = 0$;
2. $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^+} g(x) = \infty$,

then $\lim\limits_{x\to a^+}\dfrac{f(x)}{g(x)}$ exists, and

$$\lim_{x\to a^+}\frac{f(x)}{g(x)} = \lim_{x\to a^+}\frac{f'(x)}{g'(x)}\,.$$

Proof. We first prove L'Hôspital's rule for the case that $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^+} g(x) = 0$. Define $F, G : (a,b) \to \mathbb{R}$ by

$$F(x) = \begin{cases} f(x) & \text{if } x \in (a,b)\,, \\ 0 & \text{if } x = a\,, \end{cases} \qquad \text{and} \qquad G(x) = \begin{cases} g(x) & \text{if } x \in (a,b)\,, \\ 0 & \text{if } x = a\,. \end{cases}$$

Then for all $x \in (a,b)$, F, G are continuous on the closed $[a,x]$, and differentiable on the open interval with end-points (a,x). Therefore, the Cauchy Mean Value Theorem implies that there exists a point c between a and x such that

$$\frac{f'(c)}{g'(c)} = \frac{F'(c)}{G'(c)} = \frac{F(x)-F(a)}{G(x)-G(a)} = \frac{F(x)}{G(x)} = \frac{f(x)}{g(x)}\,.$$

Since c approaches a as x approaches a, we have

$$\lim_{x\to a^+}\frac{f'(c)}{g'(c)} = \lim_{c\to a^+}\frac{f'(c)}{g'(c)} = \lim_{x\to a^+}\frac{f'(x)}{g'(x)}\,;$$

thus

$$\lim_{x\to a^+}\frac{f(x)}{g(x)} = \lim_{x\to a^+}\frac{f'(c)}{g'(c)} = \lim_{x\to a^+}\frac{f'(x)}{g'(x)}\,.$$

Next we prove L'Hôspital's rule for the case that $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^+} g(x) = \infty$. Let $L = \lim\limits_{x\to a^+}\dfrac{f'(x)}{g'(x)}$ and $\varepsilon > 0$ be given. Then there exists $\delta_1 > 0$ such that

$$\Big|\frac{f'(x)}{g'(x)} - L\Big| < \frac{\varepsilon}{2} \qquad \text{whenever} \quad a < x < a + \delta_1\,(< b)\,.$$

Let $d = a + \delta_1$. For $a < x < d$, the Cauchy mean value theorem implies that for some c in (x, d) such that

$$\frac{f(x) - f(d)}{g(x) - g(d)} = \frac{f'(c)}{g'(c)}\,.$$

Note that the quotient above belongs to $\big(L - \frac{\varepsilon}{2}, L + \frac{\varepsilon}{2}\big)$ (if $a < x < d$). Moreover,

$$\begin{aligned}
\frac{f(x) - f(d)}{g(x) - g(d)} - \frac{f(x)}{g(x)} &= \frac{\big(f(x) - f(d)\big)g(x) - \big(g(x) - g(d)\big)f(x)}{\big(g(x) - g(d)\big)g(x)} \\
&= \frac{\big(f(x) - f(d)\big)g(d) - \big(g(x) - g(d)\big)f(x)}{\big(g(x) - g(d)\big)g(d)} = \frac{f'(c)}{g'(c)}\frac{g(d)}{g(x)} - \frac{f(d)}{g(x)}\,;
\end{aligned}$$

thus

$$\Big|\frac{f(x) - f(d)}{g(x) - g(d)} - \frac{f(x)}{g(x)}\Big| \leqslant \Big(|L| + \frac{\varepsilon}{2}\Big)\Big|\frac{g(d)}{g(x)}\Big| + \Big|\frac{f(d)}{g(x)}\Big| \qquad \text{whenever} \quad a < x < d\,.$$

Since $\lim\limits_{x\to a^+} g(x) = \infty$, the right-hand side of the inequality above approaches zero as x approaches a from the right. Therefore, there exists $0 < \delta < \delta_1$, such that

$$\Big|\frac{f(x) - f(d)}{g(x) - g(d)} - \frac{f(x)}{g(x)}\Big| < \frac{\varepsilon}{2} \qquad \text{whenever} \quad a < x < a + \delta\,(< d < b)\,.$$

As a consequence, if $a < x < a + \delta$,

$$\Big|\frac{f(x)}{g(x)} - L\Big| \leqslant \Big|\frac{f(x) - f(d)}{g(x) - g(d)} - \frac{f(x)}{g(x)}\Big| + \Big|\frac{f(x) - f(d)}{g(x) - g(d)} - L\Big| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which concludes the theorem. □

Remark 5.47. 1. L'Hôspital Rule can also be applied to the case when $\lim\limits_{x\to b^-}$ replaces $\lim\limits_{x\to a^+}$ in the theorem. Moreover, the one-sided limit can also be replaced by full limit $\lim\limits_{x\to c}$ if $c \in (a, b)$ (by considering L'Hôspital's Rule on (a, c) and (c, b), respectively). See Example 5.48 for more details on the full limit case.

2. L'Hôspital Rule can also be applied to limits as $x \to \infty$ or $x \to -\infty$ (and here b or a has to be changed to ∞ or $-\infty$ as well). To see this, we note that if $F(x) = f\big(\frac{1}{x}\big)$ and $G(x) = g\big(\frac{1}{x}\big)$, then either $\lim\limits_{x\to 0^+} F(x) = \lim\limits_{x\to 0^+} G(x) = 0$ or $\lim\limits_{x\to 0^+} F(x) = \lim\limits_{x\to 0^+} G(x) = \infty$; thus L'Hôspital Rule implies that

$$\lim_{x\to\infty} \frac{f'(x)}{g'(x)} = \lim_{y\to 0^+} \frac{f'\big(\frac{1}{y}\big)}{g'\big(\frac{1}{y}\big)} = \lim_{y\to 0^+} \frac{f'\big(\frac{1}{y}\big)\frac{-1}{y^2}}{g'\big(\frac{1}{y}\big)\frac{-1}{y^2}} = \lim_{y\to 0^+} \frac{F'(y)}{G'(y)} = \lim_{y\to 0^+} \frac{F(y)}{G(y)} = \lim_{x\to\infty} \frac{f(x)}{g(x)}\,.$$

3. L'Hôspital's rule only states that under suitable assumptions, if the limit of $\dfrac{f'(x)}{g'(x)}$ exists, so does the limit of $\dfrac{f(x)}{g(x)}$ and the limits are identical, but not the other way around. In other words, under the same assumptions in the statement of L'Hôspital's rule, the existence of the limit of $\dfrac{f(x)}{g(x)}$ does **NOT** implies the existence of the limit of $\dfrac{f'(x)}{g'(x)}$. For example, consider the case $f(x) = xe^{-x^{-2}}\sin(x^{-4})$ and $g(x) = e^{-x^{-2}}$. Then the Squeeze Theorem implies that $\lim\limits_{x\to 0} f(x) = \lim\limits_{x\to 0} g(x) = 0$, and
$$\lim_{x\to 0}\frac{f(x)}{g(x)} = \lim_{x\to 0} x\sin(x^{-4}) = 0\,.$$
However, since $f'(x) = \big[(1+2x^{-2})\sin(x^{-4}) - 4x^{-4}\cos(x^{-4})\big]e^{-x^{-2}}$ and $g'(x) = 2x^{-3}e^{-x^{-2}}$, we have
$$\frac{f'(x)}{g'(x)} = \frac{1}{2}(x^3+2x)\sin(x^{-4}) - \frac{2}{x}\cos(x^{-4})$$
whose limit, as x approaches 0, does not exist.

- **Indeterminate form $\dfrac{0}{0}$**

Example 5.48. Compute $\lim\limits_{x\to 0}\dfrac{e^{2x}-1}{x}$.

Let $f(x) = e^{2x} - 1$ and $g(x) = x$. Then f, g are differentiable on $(0,1)$ and $g(x) \neq 0, g'(x) \neq 0$ for all $x \in (0,1)$. Moreover,
$$\lim_{x\to 0^+}\frac{f'(x)}{g'(x)} = \lim_{x\to 0^+}\frac{2e^{2x}}{1} = 2$$
and $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} g(x) = 0$. Therefore, L'Hôspital's Rule implies that
$$\lim_{x\to 0^+}\frac{f(x)}{g(x)} = \lim_{x\to 0^+}\frac{f'(x)}{g'(x)} = 2\,.$$
Similarly, by the fact that

1. f, g are differentiable on $(-1,0)$ and $g(x) \neq 0, g'(x) \neq 0$ for all $x \in (-1,0)$,
2. $\lim\limits_{x\to 0^-}\dfrac{f'(x)}{g'(x)} = \lim\limits_{x\to 0^-}\dfrac{2e^{2x}}{1} = 2$,
3. $\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0^+} g(x) = 0$,

L'Hôspital's Rule implies that $\displaystyle\lim_{x\to 0^-}\frac{f(x)}{g(x)} = \lim_{x\to 0^-}\frac{f'(x)}{g'(x)} = 2$. Theorem **??** then shows that $\displaystyle\lim_{x\to 0}\frac{f(x)}{g(x)} = 2$ exists.

From the discussion in Example 5.48, using L'Hôspital's Rule in Theorem 5.46 we deduce the following L'Hôspital's Rule for the full limit case.

Theorem 5.46*

Let $a < c < b$, and f, g be differentiable functions on $(a,b)\backslash\{c\}$. Assume that $g'(x) \neq 0$ for all $x \in (a,b)\backslash\{c\}$. If the limit of $\dfrac{f(x)}{g(x)}$ as x approaches c produces the indeterminate form $\dfrac{0}{0}$ $\big($or $\dfrac{\infty}{\infty}\big)$; that is, $\displaystyle\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = 0$ $\big($or $\displaystyle\lim_{x\to c} f(x) = \lim_{x\to c} g(x) = \infty\big)$, then

$$\lim_{x\to c}\frac{f(x)}{g(x)} = \lim_{x\to c}\frac{f'(x)}{g'(x)}$$

provided the limit on the right exists.

- **Indeterminate form** $\dfrac{\infty}{\infty}$

Example 5.49. In this example we compute $\displaystyle\lim_{x\to\infty}\frac{\ln x}{x}$. Note that $\displaystyle\lim_{x\to\infty}\frac{\frac{d}{dx}\ln x}{\frac{d}{dx}x} = \lim_{x\to\infty}\frac{1}{x} = 0$, so L'Hôspital's Rule implies that

$$\lim_{x\to\infty}\frac{\ln x}{x} = \lim_{x\to\infty}\frac{\frac{d}{dx}\ln x}{\frac{d}{dx}x} = 0\,.$$

In fact, the logarithmic function $y = \ln x$ grows slower than any power function; that is,

$$\lim_{x\to\infty}\frac{\ln x}{x^p} = 0 \qquad \forall\, p > 0\,.$$

To see this, note that $\displaystyle\lim_{x\to\infty}\frac{\frac{d}{dx}\ln x}{\frac{d}{dx}x^p} = \lim_{x\to\infty}\frac{\frac{1}{x}}{px^{p-1}} = \frac{1}{p}\lim_{x\to\infty}\frac{1}{x^p} = 0$, so L'Hôspital's Rule implies that

$$\lim_{x\to\infty}\frac{\ln x}{x^p} = \lim_{x\to\infty}\frac{\frac{d}{dx}\ln x}{\frac{d}{dx}x^p} = 0\,.$$

- **Indeterminate form** $0\cdot\infty$

Example 5.50. Compute $\displaystyle\lim_{x\to\infty} e^{-x}\sqrt{x}$. Rewrite $e^{-x}\sqrt{x}$ as $\dfrac{\sqrt{x}}{e^x}$ and note that

$$\lim_{x\to\infty}\frac{\frac{d}{dx}\sqrt{x}}{\frac{d}{dx}e^x} = \lim_{x\to\infty}\frac{\frac{1}{2\sqrt{x}}}{e^x} = \lim_{x\to\infty}\frac{1}{2\sqrt{x}e^x} = 0\,.$$

Therefore, L'Hôspital's Rule implies that

$$\lim_{x\to\infty}\frac{\sqrt{x}}{e^x}=\lim_{x\to\infty}\frac{\frac{d}{dx}\sqrt{x}}{\frac{d}{dx}e^x}=0\,.$$

In fact, the natural exponential function $y=e^x$ grows faster than any power function; that is,

$$\lim_{x\to\infty}\frac{x^p}{e^x}=0\qquad\forall\,p>0\,.$$

The proof is left as an exercise.

- **Indeterminate form** 1^∞

Example 5.51. In this example we compute $\lim\limits_{x\to0}(1+x)^{\frac{1}{x}}$. Rewrite $(1+x)^{\frac{1}{x}}$ as $e^{\frac{\ln(1+x)}{x}}$. If the limit $\lim\limits_{x\to0}\frac{\ln(1+x)}{x}$ exists, then the continuity of the exponential function implies that

$$\lim_{x\to0}(1+x)^{\frac{1}{x}}=\exp\big(\lim_{x\to0}\frac{\ln(1+x)}{x}\big)\,.$$

Nevertheless, since $\lim\limits_{x\to0}\ln(1+x)=0$, $\lim\limits_{x\to0}x=0$ and

$$\lim_{x\to0}\frac{\frac{d}{dx}\ln(1+x)}{\frac{d}{dx}x}=\lim_{x\to0}\frac{1}{1+x}=1$$

L'Hôspital's Rule implies that

$$\lim_{x\to0}\frac{\ln(1+x)}{x}=\lim_{x\to0}\frac{\frac{d}{dx}\ln(1+x)}{\frac{d}{dx}x}=1\,;$$

thus $\lim\limits_{x\to0}(1+x)^{\frac{1}{x}}=\exp(1)=e$.

- **Indeterminate form** 0^0

Example 5.52. In this example we compute $\lim\limits_{x\to0^+}(\sin x)^x$. When $\sin x>0$, we have

$$(\sin x)^x=e^{x\ln\sin x}=e^{\frac{\ln\sin x}{1/x}}\,.$$

Since

$$\lim_{x\to0^+}\frac{\frac{d}{dx}\ln\sin x}{\frac{d}{dx}\frac{1}{x}}=\lim_{x\to0^+}\frac{\frac{\cos x}{\sin x}}{-\frac{1}{x^2}}=-\lim_{x\to0^+}\frac{x}{\sin x}x\cos x=0\,,$$

by L'Hôspital's Rule and the continuity of the natural exponential function we find that

$$\lim_{x\to0^+}(\sin x)^x=\lim_{x\to0^+}e^{\frac{\ln\sin x}{1/x}}=e^0=1\,.$$

• **Indeterminate form $\infty - \infty$**

Example 5.53. Compute $\lim\limits_{x\to 1+}\Big(\dfrac{1}{\ln x}-\dfrac{1}{x-1}\Big)$.

Rewrite $\dfrac{1}{\ln x}-\dfrac{1}{x-1}=\dfrac{x-1-\ln x}{(x-1)\ln x}$ and note that the right-hand side produces indeterminate form $\dfrac{0}{0}$ as x approaches from the right. Also note that

$$\frac{\frac{d}{dx}(x-1-\ln x)}{\frac{d}{dx}(x-1)\ln x}=\frac{1-\frac{1}{x}}{\ln x+\frac{x-1}{x}}=\frac{x-1}{x\ln x+x-1}$$

which, as x approaches 1 from the right, again produces indeterminate form $\dfrac{0}{0}$. In order to find the limit of the right-hand side we compute

$$\lim_{x\to 1^+}\frac{\frac{d}{dx}(x-1)}{\frac{d}{dx}(x\ln x+x-1)}=\lim_{x\to 1^+}\frac{1}{\ln x+1+1}=\frac{1}{2}\,;$$

thus L'Hôspital's Rule implies that

$$\lim_{x\to 1^+}\frac{x-1}{x\ln x+x-1}=\lim_{x\to 1^+}\frac{\frac{d}{dx}(x-1)}{\frac{d}{dx}(x\ln x+x-1)}=\frac{1}{2}\,.$$

This in turm shows that

$$\lim_{x\to 1^+}\frac{x-1-\ln x}{(x-1)\ln x}=\lim_{x\to 1^+}\frac{\frac{d}{dx}(x-1-\ln x)}{\frac{d}{dx}(x-1)\ln x}=\lim_{x\to 1^+}\frac{x-1}{x\ln x+x-1}=\frac{1}{2}\,.$$

5.7 The Inverse Trigonometric Functions: Differentiation

Definition 5.54

The arcsin, arccos, and arctan functions are the inverse functions of the function $f:\big[-\frac{\pi}{2},\frac{\pi}{2}\big]\to\mathbb{R}$, $g:[0,\pi]\to\mathbb{R}$, and $h:\big(-\frac{\pi}{2},\frac{\pi}{2}\big)\to\mathbb{R}$, respectively, where $f(x)=\sin x$, $g(x)=\cos x$ and $h(x)=\tan x$. In other words,

1. $y=\arcsin x$ if and only if $\sin y=x$, where $-\dfrac{\pi}{2}\leqslant y\leqslant\dfrac{\pi}{2}$, $-1\leqslant x\leqslant 1$.
2. $y=\arccos x$ if and only if $\cos y=x$, where $0\leqslant y\leqslant\pi$, $-1\leqslant x\leqslant 1$.
3. $y=\arctan x$ if and only if $\tan y=x$, where $-\dfrac{\pi}{2}<y<\dfrac{\pi}{2}$, $-\infty<x<\infty$.

Remark 5.55. Since arcsin, arccos and arctan look like the inverse function of sin, cos and tan, respectively, often times we also write arcsin as $\sin^{-1}$, arccos as $\cos^{-1}$, and arctan as $\tan^{-1}$.

Example 5.56. $\arcsin\frac{1}{2} = \frac{\pi}{6}$, $\arccos\big(\frac{-\sqrt{2}}{2}\big) = \frac{3\pi}{4}$, and $\arctan 1 = \frac{\pi}{4}$.

Example 5.57. Suppose that $y = \arcsin x$. Then $y \in \big[-\frac{\pi}{2}, \frac{\pi}{2}\big]$ which implies that $\cos y \geqslant 0$. Therefore, by the fact that $\sin^2 y + \cos^2 y = 1$, we have

$$\cos y = \sqrt{1 - \sin^2 y} = \sqrt{1 - x^2} \qquad \text{if} \quad y = \arcsin x\,.$$

In other words, $\cos(\arcsin x) = \sqrt{1 - x^2}$.

Similarly, if $y = \arccos x$, then $y \in (0, \pi)$ which implies that $\sin y \geqslant 0$. Therefore,

$$\sin y = \sqrt{1 - \cos^2 y} = \sqrt{1 - x^2} \qquad \text{if} \quad y = \arccos x$$

or equivalently, $\sin(\arccos x) = \sqrt{1 - x^2}$.

Example 5.58. Suppose that $y = \arctan x$ for some $x \in \mathbb{R}$. Then $y \in \big(-\frac{\pi}{2}, \frac{\pi}{2}\big)$ which implies that $\cos y > 0$. Therefore,

$$\cos y = \frac{1}{\sec y} = \frac{1}{\sqrt{1 + \tan^2 y}} = \frac{1}{\sqrt{1 + x^2}}\,.$$

As for $\sin y$, we note that $y > 0$ if and only if $x > 0$; thus $\sin y = \frac{x}{\sqrt{1 + x^2}}$ $\Big($instead of $\frac{-x}{\sqrt{1 + x^2}}\Big)$. Therefore,

$$\sin(\arctan x) = \frac{x}{\sqrt{1 + x^2}} \quad \text{and} \quad \cos(\arctan x) = \frac{1}{\sqrt{1 + x^2}}\,.$$

Theorem 5.59: Differentiation of Inverse Trigonometric Functions

1. $\frac{d}{dx}\arcsin x = \frac{1}{\sqrt{1 - x^2}}$ for all $-1 < x < 1$.

2. $\frac{d}{dx}\arccos x = -\frac{1}{\sqrt{1 - x^2}}$ for all $-1 < x < 1$.

3. $\frac{d}{dx}\arctan x = \frac{1}{1 + x^2}$ for all $x \in \mathbb{R}$.

Proof. By Inverse Function Differentiation,

$$\frac{d}{dx}\arcsin x = \frac{1}{\cos(\arcsin x)} = \frac{1}{\sqrt{1-x^2}} \qquad \forall\, x \in (-1,1)\,,$$
$$\frac{d}{dx}\arccos x = \frac{1}{-\sin(\arccos x)} = -\frac{1}{\sqrt{1-x^2}} \qquad \forall\, x \in (-1,1)\,,$$

and

$$\frac{d}{dx}\arctan x = \frac{1}{\sec^2(\arctan x)} = \frac{1}{1+\tan^2(\arctan x)} = \frac{1}{1+x^2} \qquad \forall\, x \in \mathbb{R}\,. \qquad \square$$

Remark 5.60. By Theorem 5.59,

$$\frac{d}{dx}\big(\arcsin x + \arccos x\big) = \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-x^2}} = 0 \qquad \forall\, -1 < x < 1\,.$$

Therefore, the function $y = \arcsin x + \arccos x$ is constant on the interval $(-1,1)$. The constant can be obtained by testing with $x = 0$ and we find that

$$\arcsin x + \arccos x = \frac{\pi}{2} \qquad \forall\, x \in [-1,1]\,, \tag{5.7.1}$$

where the value of the left-hand side at $x = \pm 1$ are computed separately.

Example 5.61. Find the derivative of $y = \arcsin x + x\sqrt{1-x^2}$.

By Theorem 5.59 and the chain rule, for $-1 < x < 1$ we have

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} + \sqrt{1-x^2} - x\cdot\frac{1}{2}(1-x^2)^{-\frac{1}{2}}(2x) = 2\sqrt{1-x^2}\,.$$

Example 5.62. Find the derivative of $y = \arctan\sqrt{x}$.

By the chain rule,

$$\frac{dy}{dx} = \frac{1}{1+\sqrt{x}^2}\frac{d}{dx}\sqrt{x} = \frac{1}{1+x}\frac{1}{2\sqrt{x}} = \frac{1}{2\sqrt{x}(1+x)}\,.$$

5.8 Inverse Trigonometric Functions: Integration

Theorem 5.63

Let a be a positive real number. Then

1. $\displaystyle\int \frac{dx}{\sqrt{a^2-x^2}} = \arcsin\frac{x}{a} + C.$ 2. $\displaystyle\int \frac{dx}{a^2+x^2} = \frac{1}{a}\arctan\frac{x}{a} + C.$

Proof. 1. Let $x = a\sin u$. Then $dx = a\cos u du$; thus

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \int \frac{a\cos u}{\sqrt{a^2(1-\sin^2 u)}}\, du = \int du = u + C = \arcsin\frac{x}{a} + C\,.$$

2. Let $x = a\tan u$. Then $dx = a\sec^2 u du$; thus

$$\int \frac{dx}{a^2+x^2} = \int \frac{a\sec^2 u}{a^2(1+\tan^2 u)}\, du = \frac{1}{a}\int du = \frac{u}{a} + C = \frac{1}{a}\arctan\frac{x}{a} + C\,. \qquad \square$$

Example 5.64. Find the indefinite integral $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}}$, where $a > 0$ is a constant.

Let $x = a\sec u$. Then $dx = a\sec u\tan u du$; thus

$$\begin{aligned}\int \frac{dx}{\sqrt{x^2-a^2}} &= \int \frac{a\sec u\tan u}{\sqrt{a^2(\sec^2 u-1)}}\, du = \int \sec u\, du = \ln|\sec u + \tan u| + C \\ &= \ln\Big|\frac{x}{a} + \frac{\sqrt{x^2-a^2}}{a}\Big| + C = \ln\big|x + \sqrt{x^2-a^2}\big| + C\,.\end{aligned}$$

Example 5.65. Find the indefinite integral $\displaystyle\int \frac{dx}{\sqrt{x^2+a^2}}$, where $a > 0$ is a constant.

Let $x = a\tan u$. Then $dx = a\sec^2 u du$; thus

$$\begin{aligned}\int \frac{dx}{\sqrt{a^2+x^2}} &= \int \frac{a\sec^2 u}{\sqrt{a^2(\tan^2 u+1)}}\, du = \int \sec u\, du = \ln|\sec u + \tan u| + C \\ &= \ln\Big|\frac{\sqrt{x^2+a^2}}{a} + \frac{x}{a}\Big| + C = \ln\big|x + \sqrt{x^2+a^2}\big| + C\,.\end{aligned}$$

Example 5.66. Find the indefinite integral $\displaystyle\int \frac{dx}{x\sqrt{x^2-a^2}}$, where $a > 0$ is a constant.

Let $x = a\sec u$. Then $dx = a\sec u\tan u$; thus

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \int \frac{a\sec u\tan u}{a\sec u\sqrt{a^2(\sec^2 u-1)}} = \frac{1}{a}\int du = \frac{u}{a} + C\,.$$

If $x = a\sec u$, then $\tan u = \dfrac{\sqrt{x^2-a^2}}{a}$; thus $u = \arctan\dfrac{\sqrt{x^2-a^2}}{a}$ which implies that

$$\int \frac{dx}{x\sqrt{x^2-a^2}} = \frac{1}{a}\arctan\frac{\sqrt{x^2-a^2}}{a} + C\,.$$

Example 5.67. Find the indefinite integral $\displaystyle\int \frac{dx}{\sqrt{e^{2x}-1}}$.

Let $u = e^x$. Then $du = e^x dx$; thus $dx = \dfrac{du}{u}$ which implies that

$$\int \frac{dx}{\sqrt{e^{2x}-1}} = \int \frac{du}{u\sqrt{u^2-1}} = \arctan\sqrt{u^2-1} + C = \arctan\sqrt{e^{2x}-1} + C\,.$$

Example 5.68. Find the indefinite integral $\displaystyle\int \frac{x+2}{\sqrt{4-x^2}}\,dx$.

Let $x = 2\sin u$. Then $dx = 2\cos u du$; thus

$$\begin{aligned}\int \frac{x+2}{\sqrt{4-x^2}}\,dx &= \int \frac{2\sin u + 2}{\sqrt{4-4\sin^2 u}} \cdot 2\cos u\,du = \int (2\sin u + 2)\,du = 2u - 2\cos u + C \\ &= 2\arcsin\frac{x}{2} - 2\sqrt{1-\big(\frac{x}{2}\big)^2} + C = 2\arcsin\frac{x}{2} - \sqrt{4-x^2} + C\,.\end{aligned}$$

Example 5.69. Find the indefinite integral $\displaystyle\int \frac{dx}{x^2-4x+7}$.

First we complete the square and obtain that $x^2 - 4x + 7 = (x-2)^2 + 3$. Let $x - 2 = \sqrt{3}\tan u$. Then $dx = \sqrt{3}\sec^2 u du$; thus

$$\int \frac{dx}{x^2-4x+7} = \int \frac{\sqrt{3}\sec^2 u}{3\tan^2 u + 3}\,du = \frac{1}{\sqrt{3}}\int du = \frac{1}{\sqrt{3}}u + C = \frac{1}{\sqrt{3}}\arctan\frac{x-2}{\sqrt{3}} + C\,.$$

Example 5.70. Find the indefinite integral $\displaystyle\int \sqrt{\frac{1-x}{1+x}}\,dx$.

Note that the integrand can be rewritten as $\dfrac{1-x}{\sqrt{1-x^2}}$. Therefore,

$$\begin{aligned}\int \sqrt{\frac{1-x}{1+x}}\,dx &= \int \frac{1-x}{\sqrt{1-x^2}}\,dx = \int \frac{1}{\sqrt{1-x^2}}\,dx - \int \frac{x}{\sqrt{1-x^2}}\,dx \\ &= \arcsin x + \sqrt{1-x^2} + C\,.\end{aligned}$$

Example 5.71. In this example, we compute $\displaystyle\int \arcsin x\,dx$. Note the by the substitution $x = \sin u$,

$$\int \arcsin x\,dx = \int u\cos u\,du\,;$$

thus it suffices to compute the anti-derivative of the function $y = x\cos x$. We first compute the definite integral $\displaystyle\int_0^a x\cos x\,dx$.

By Example 4.12, for $0 < x < \pi$ we have

$$\sum_{i=1}^{n}\sin(ix) = \frac{1}{2\sin\frac{x}{2}}\Big[\cos\frac{x}{2} - \cos\big((n+\frac{1}{2})x\big)\Big]\,.$$

Therefore, if $0 < x < \pi$,

$$\begin{aligned}
\sum_{i=1}^{n} i\cos(ix) &= \frac{d}{dx}\sum_{i=1}^{n}\sin(ix) = \frac{d}{dx}\frac{1}{2\sin\frac{x}{2}}\Big[\cos\frac{x}{2} - \cos\big((n+\frac{1}{2})x\big)\Big] \\
&= \frac{-\cos\frac{x}{2}}{4\sin^2\frac{x}{2}}\Big[\cos\frac{x}{2} - \cos\big((n+\frac{1}{2})x\big)\Big] \\
&\quad + \frac{1}{2\sin\frac{x}{2}}\Big[-\frac{1}{2}\sin\frac{x}{2} + (n+\frac{1}{2})\sin\big((n+\frac{1}{2})x\big)\Big].
\end{aligned}$$

By partitioning $[0, a]$ into n sub-intervals with equal length, the Riemann sum of $y = x\cos x$ for this partition given by the right end-point rule is

$$I_n = \sum_{i=1}^{n}\frac{ia}{n}\cos\frac{ia}{n}\frac{a}{n} = \frac{a^2}{n^2}\sum_{i=1}^{n} i\cos\frac{ia}{n}.$$

Letting $r = \frac{a}{2n}$, we find that

$$\begin{aligned}
I_n &= 4r^2\sum_{i=1}^{n} i\cos(2ir) \\
&= \frac{-r^2\cos r}{\sin^2 r}\Big[\cos r - \cos(a+r)\Big] + \frac{r}{\sin r}\Big[-r\sin r + (a+r)\sin(a+r)\Big]
\end{aligned}$$

which, by the fact that $\frac{\sin x}{x} \to 1$ as $x \to 0$ and $r \to 0$ as $n \to \infty$, implies that

$$\int_0^a x\cos x\,dx = \lim_{n\to\infty} I_n = -(1-\cos a) + a\sin a = a\sin a + \cos a - 1.$$

The identity above further implies that

$$\int x\cos x\,dx = x\sin x + \cos x + C;$$

thus with the substitution $x = \sin u$,

$$\int \arcsin x\,dx = \int u\cos u\,du = u\sin u + \cos u + C = x\arcsin x + \sqrt{1-x^2} + C.$$

Using (5.7.1), we also find that

$$\begin{aligned}
\int \arccos x\,dx &= \int \big(\frac{\pi}{2} - \arcsin x\big)\,dx = \frac{\pi}{2}x - x\arcsin x - \sqrt{1-x^2} + C \\
&= x\big(\frac{\pi}{2} - \arcsin x\big) - \sqrt{1-x^2} + C = x\arccos x - \sqrt{1-x^2} + C.
\end{aligned}$$