Calculus 微積分

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Chapter 0

Preliminary

0.1 Functions and Their Graphs

Definition 0.1: Real-Valued Functions of a Real Variable

Let $X, Y \subseteq \mathbb{R}$ be subsets of real numbers. A real-valued function f of a real variable x from X to Y is a correspondence that assigns to each element x in X exactly one number y in Y. Here X is called the domain of f and is usually denoted by $\mathrm{Dom}(f)$, Y is called "the" co-domain of f, the number y is called the image of x under f and is usually denoted by $f(x)$, which is called the value of f at x. The range of f, denoted by $\mathrm{Ran}(f)$, is a subset of Y consisting of all images of numbers in X. In other words,

$$\mathrm{Ran}(f) \equiv \text{the range of } f \equiv \left\{ f(x) \,\middle|\, x \in X \right\}.$$

Remark 0.2. Given a way of assignment $x \mapsto f(x)$ without specifying where x is chosen from, we still treat f as a function and $\mathrm{Dom}(f)$ is considered as the collection of all $x \in \mathbb{R}$ such that $f(x)$ is well-defined. For example, $f(x) = x + 1$ and $g(x) = \dfrac{x^2 - 1}{x - 1}$ are both considered as functions with

$$\mathrm{Dom}(f) = \mathbb{R} \qquad \text{and} \qquad \mathrm{Dom}(g) = \mathbb{R}\backslash\{1\}.$$

Since $\mathrm{Dom}(f) \neq \mathrm{Dom}(g)$, f and g are considered as different functions even though $f(x) = g(x)$ for all $x \neq 1$.

Terminologies:

1. Explicit form of a function: $y = f(x)$;

2. Implicit form of a function: $F(x, y) = 0$.（参考影片）

Definition 0.3

A function f is a polynomial function if f takes the form

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0 \,,$$

where $a_0, a_1, a_2, \cdots, a_n$ are real numbers, called coefficients of the polynomial, and n is a non-negative integer. If $a_n \neq 0$, then a_n is called the leading coefficient, and n is called the degree of the polynomial. A rational function is the quotient of two polynomials.

Definition 0.4

The graph of the function $y = f(x)$ consists of all points $\big(x, f(x)\big)$, where x is in the domain of f. In other words,

$$\mathrm{G}(f) \equiv \text{the graph of } f \equiv \Big\{\big(x, f(x)\big) \,\Big|\, x \in \mathrm{Dom}(f)\Big\} \,.$$

Definition 0.5: Composite Functions

Let f and g be functions. The function $f \circ g$, read f circle g, is the function defined by $(f \circ g)(x) = f\big(g(x)\big)$. The domain of $f \circ g$ is the set of all x in the domain of g such that $g(x)$ is in the domain of f. In other words,

$$\mathrm{Dom}(f \circ g) = \big\{x \in \mathrm{Dom}(g) \,\big|\, g(x) \in \mathrm{Dom}(f)\big\} \,.$$

0.2 Trigonometric Functions

Definition 0.6

An angle consists of an initial ray, a terminal ray and a vertex where two rays intersects. An angle is in standard position when its initial ray coincides with the positive x-axis and its vertex is at the origin. Positive angles are measured counterclockwise, and negative angles are measured clockwise.

Let θ be a central angle of a circle of radius 1. The radian measure of θ is defined to be the length of the arc of the sector.

Remark 0.7. Using radian measure of θ, the length s of a circular arc of radius r is given by $s = r\theta$.

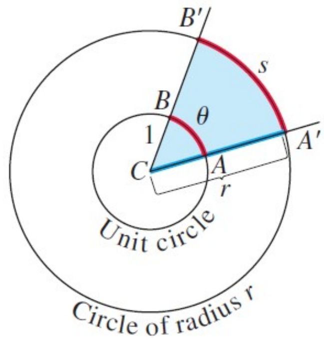

Figure 1: The radian measure of the central angle $A'CB'$ is the number $u = s/r$. For a unit circle of radius $r = 1$, u is the length of arc AB that central angle ACB cuts from the unit circle.

Remark 0.8. For a point P on the plane with Cartesian coordinate (x, y), let $r = \sqrt{x^2 + y^2}$ and θ be the angle in standard position with $\overrightarrow{OP}$ as the terminal ray. The ordered pair (r, θ) is called the ***polar coordinate*** of the point P.

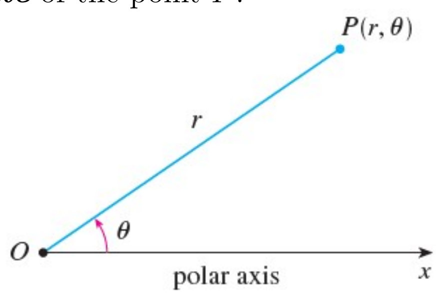

Figure 2: Polar coordinate

Definition 0.9

Let θ be an angle in standard position, and the terminal ray intersects the circle centered at the origin of radius r at point (x, y). The trigonometric functions sine, cosine, tangent, cotangent, secant and cosecant, abbreviated as sin, cos, tan, cot, sec and csc, respectively, of angle θ are defined by

$$\sin\theta = \frac{y}{r}, \quad \cos\theta = \frac{x}{r}, \quad \tan\theta = \frac{y}{x}, \quad \cot\theta = \frac{x}{y}, \quad \sec\theta = \frac{r}{x} \quad \text{and} \quad \csc\theta = \frac{r}{y},$$

provided that the quotients make sense.

Remark 0.10. Suppose that a point P has polar coordinate (r, θ). Then the Cartesian coordinate of P is $(r\cos\theta, r\sin\theta)$.

Proposition 0.11: Properties of Trigonometric Functions

1. For all real numbers θ,

$$\sin^2\theta + \cos^2\theta = 1\,, \quad 1 + \tan^2\theta = \sec^2\theta\,, \quad 1 + \cot^2\theta = \csc^2\theta\,.$$

2. For all real numbers θ,

$$\begin{aligned} &\sin(-\theta) = -\sin\theta\,, \quad \cos(-\theta) = \cos\theta\,, \quad \tan(-\theta) = -\tan\theta\,, \\ &\cot(-\theta) = -\cot\theta\,, \quad \sec(-\theta) = \sec\theta\,, \quad \csc(-\theta) = -\csc\theta\,. \end{aligned}$$

3. For all real numbers θ,

$$\begin{aligned} &\sin\Big(\theta + \frac{\pi}{2}\Big) = \cos\theta\,, \quad \cos\Big(\theta + \frac{\pi}{2}\Big) = -\sin\theta\,, \quad \tan\Big(\theta + \frac{\pi}{2}\Big) = -\cot\theta\,, \\ &\sin(\theta + \pi) = -\sin\theta\,, \quad \cos(\theta + \pi) = -\cos\theta\,, \quad \tan(\theta + \pi) = \tan\theta\,. \end{aligned}$$

4. (**Law of Cosines**): Let a, b, c be the length of sides of a triangle, and θ be the angle opposite to the side with length c. Then $c^2 = a^2 + b^2 - 2ab\cos\theta$.

5. (**Sum and Difference Formulas**): Let θ, ϕ be real numbers. Then

$$\sin(\theta \pm \phi) = \sin\theta\cos\phi \pm \sin\phi\cos\theta\,, \quad \cos(\theta \pm \phi) = \cos\theta\cos\phi \mp \sin\theta\sin\phi\,.$$

6. (**Double-Angle Formulas**): For all real numbers θ,

$$\sin(2\theta) = 2\sin\theta\cos\theta\,, \quad \cos(2\theta) = 2\cos^2\theta - 1 = 1 - 2\sin^2\theta\,.$$

7. (**Half-Angle Formulas**): For all real numbers θ,

$$\cos^2\frac{\theta}{2} = \frac{1+\cos\theta}{2}\,, \quad \sin^2\frac{\theta}{2} = \frac{1-\cos\theta}{2}\,, \quad \tan\frac{\theta}{2} = \frac{\sin\theta}{1+\cos\theta}\,.$$

8. (**Triple-Angle Formulas**): For all real numbers θ,

$$\cos(3\theta) = 4\cos^3\theta - 3\cos\theta\,, \quad \sin(3\theta) = 3\sin\theta - 4\sin^3\theta\,.$$

9. (**Sum-to-Product Formulas**): For all real numbers θ and ϕ,

$$\begin{aligned} &\sin\theta + \sin\phi = 2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}\,, \quad \sin\theta - \sin\phi = 2\sin\frac{\theta-\phi}{2}\cos\frac{\theta+\phi}{2}\,, \\ &\cos\theta + \cos\phi = 2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}\,, \quad \cos\theta - \cos\phi = 2\sin\frac{\theta+\phi}{2}\sin\frac{\phi-\theta}{2}\,. \end{aligned}$$

Theorem 0.12: de Moivre (棣美弗)

For each real number θ and natural number n,

$$(\cos\theta + i\sin\theta)^n = \cos(n\theta) + i\sin(n\theta)\,. \tag{0.2.1}$$

Proof. Clearly (0.2.1) holds for $n = 1$. Suppose that (0.2.1) holds for $n = k$ for some natural number k. Then by the sum and difference formulas,

$$\begin{aligned}(\cos\theta + i\sin\theta)^{k+1} &= (\cos\theta + i\sin\theta)^k \cdot (\cos\theta + i\sin\theta)\\ &= \big[\cos(k\theta) + i\sin(k\theta)\big] \cdot (\cos\theta + i\sin\theta)\\ &= \cos(k\theta)\cos\theta - \sin(k\theta)\sin\theta + i\big[\sin(k\theta)\cos\theta + \cos(k\theta)\sin\theta\big]\\ &= \cos[(k+1)\theta] + i\sin[(k+1)\theta]\end{aligned}$$

which shows that (0.2.1) holds for $n = k + 1$. By induction, we find that (0.2.1) holds for all natural number n. $\square$

Theorem 0.13

Let θ be a real number and $0 \leqslant \theta < \dfrac{\pi}{2}$. Then

$$\sin\theta \leqslant \theta \leqslant \tan\theta\,. \tag{0.2.2}$$

Proof. Inequality (0.2.2) follows from the following figure

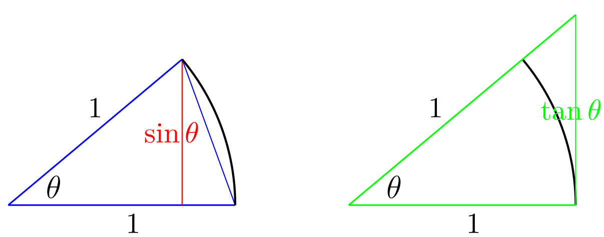

Figure 3: The area of the sector is larger than the area of the blue triangle but is smaller than the green triangle

which shows $\dfrac{1}{2}\sin\theta \leqslant \dfrac{1}{2}\theta \leqslant \dfrac{1}{2}\tan\theta$. $\square$

Chapter 1

Limits and Continuity

1.1 Limits of Functions

Goal: Given a function f defined "near c", find the value of f at x when x is "arbitrarily close" to c. (給定一函數 f，我們想知道「當除 c 之外的點到 c 的距離愈來愈近時，其函數值是否向某數集中」)

Notation: When there exists such a value, the value is denoted by $\lim\limits_{x\to c} f(x)$.

Example 1.1. Consider the function $g(x) = \dfrac{x^2-1}{x-1}$ given in Remark 0.2, and

$$h(x) = \begin{cases} \dfrac{x^2-1}{x-1} & \text{if } x \neq 1\,, \\ 0 & \text{if } x = 1\,. \end{cases}$$

Then the limit of g at 1 should be the same as the limit of h at 1. Therefore, to consider the limit of a function at a point c, the value of the function at c is not important at all.

Example 1.2. Let $g(x) = \dfrac{x^2-1}{x-1}$. Then $\mathrm{Dom}(g) = \mathbb{R}\backslash\{1\}$ and $g(x) = x+1$ if $x \neq 1$. Therefore, the graph of g is given by

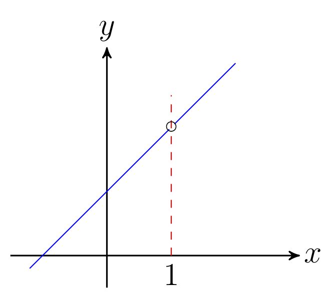

Figure 1.1: The graph of function $g(x) = \dfrac{x^2-1}{x-1}$

Then (by looking at the graph of g we find that) $\lim_{x\to 1} g(x) = 2$.

Example 1.3. Let $f(x) = \begin{cases} 1 & \text{if } x \neq 2\,, \\ 0 & \text{if } x = 2\,. \end{cases}$ The graph of f is given by

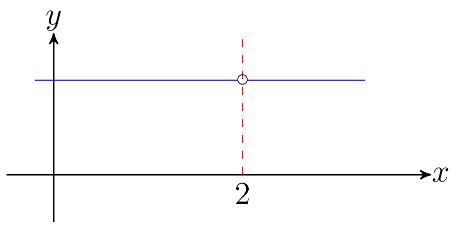

Figure 1.2: The graph of function $f(x)$

Then (by looking at the graph of f we find that) $\lim_{x\to 2} f(x) = 1$.

Next we give some examples in which the limit of functions (at certain points) do not exist.

Example 1.4. （詳見影片）Let $f(x) = \sin\frac{1}{x}$. Then $\mathrm{Dom}(f) = \mathbb{R}\backslash\{0\}$. For the graph of f, we note that if $x \in I_n \equiv \left[\frac{1}{2n\pi + 2\pi}, \frac{1}{2n\pi}\right]$ for some $n \in \mathbb{N}$, the graph of f on I_n must touch $x = 1$ and $x = -1$ once. Therefore, the graph of f looks like

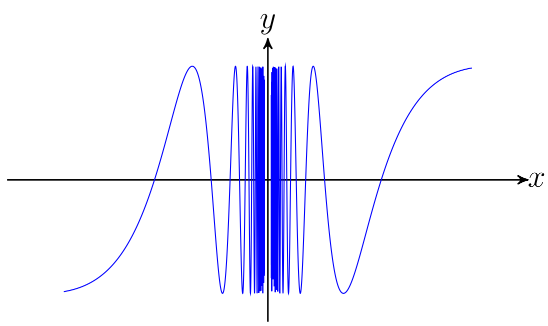

Figure 1.3: The graph of function $f(x) = \sin\frac{1}{x}$

In any interval containing 0, there are infinitely many points whose image under f is 1, and there are always infinitely many points whose image under f is -1. In fact, in any interval containing 0 and $L \in [-1, 1]$ there are infinitely many points whose image under f is L. Therefore, $\lim_{x\to 0} f(x)$ D.N.E. (does not exist).

Example 1.5. Let $f(x) = \frac{|x|}{x}$. Then $f(x) = 1$ if $x > 0$, $f(x) = -1$ if $x < 0$, and the graph of f is given by

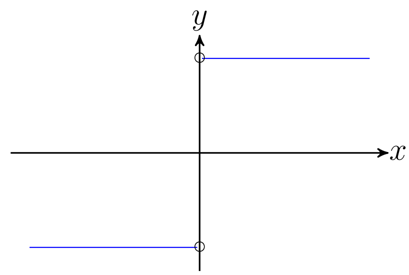

Figure 1.4: The graph of function $f(x) = \dfrac{|x|}{x}$

By observation (that is, looking at the graph of f), $\lim\limits_{x\to 0} f(x)$ D.N.E.

Example 1.6. （詳見影片）Consider the Dirichlet function

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}\,, \\ 1 & \text{if } x \notin \mathbb{Q}\,, \end{cases}$$

where $\mathbb{Q}$ denotes the collection of rational numbers（有理數）. Then $\lim\limits_{x\to c} f(x)$ D.N.E. for all c.

Example 1.7. （詳見影片）Let $f : (0,\infty) \to \mathbb{R}$ be given by

$$f(x) = \begin{cases} \dfrac{1}{p} & \text{if } x = \dfrac{q}{p}, \text{ where } p, q \in \mathbb{N} \text{ and } (p,q) = 1\,, \\ 0 & \text{if } x \text{ is irrational（無理數）.} \end{cases}$$

Then $\lim\limits_{x\to c} f(x) = 0$ for all $c \in (0,\infty)$.

Definition 1.8

Let f be a function defined on an open interval containing c (except possibly at c), and L be a real number. The statement

$$\lim_{x\to c} f(x) = L\,, \quad \text{read "the limit of } f \text{ at } c \text{ is } L\text{"},$$

means that for every $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\big|f(x) - L\big| < \varepsilon \quad \text{whenever} 0 < |x - c| < \delta\,.$$

Explanation:（詳見影片）因為 $|f(x) - L| < \varepsilon$ 等價於 $f(x) \in (L-\varepsilon, L+\varepsilon)$，所以定義敘述中的 ε 可視為用來度量 $f(x)$ 向 L 這個數集中的程度。定義所述是指對於任意給定的集中程度 $\varepsilon > 0$，一定可以找到在 c 附近的一個範圍（以到 c 的距離小於 δ 來表示），滿足

此範圍中的點之函數值落入想要其落入的集中區域 $(L-\varepsilon, L+\varepsilon)$ 之內。此即「當除 c 之外的點到 c 的距離愈來愈近時，其函數值向 L 集中」的意思。

Example 1.9. In this example we show that $\lim\limits_{x\to 1}(x+1)=2$ using Definition 1.8.

Let $\varepsilon>0$ be given. Define $\delta=\varepsilon$. Then $\delta>0$ and if $0<|x-1|<\delta$, we have

$$|(x+1)-2| = |x-1| < \delta = \varepsilon\,.$$

One could also pick $\delta=\dfrac{\varepsilon}{2}$ so that if $0<|x-1|<\delta$,

$$|(x+1)-2| = |x-1| < \delta = \frac{\varepsilon}{2} < \varepsilon\,.$$

Example 1.10. Show that $\lim\limits_{x\to 2}x^2=4$. If $\varepsilon=1$, we can choose $\delta=\min\big\{\sqrt{5}-2, 2-\sqrt{3}\big\}$ so that $\delta>0$ and if $0<|x-2|<\delta$ we must have $|x^2-4|<1$.

For general ε, we can choose $\delta=\min\big\{\sqrt{4+\varepsilon}-2, 2-\sqrt{4-\varepsilon}\big\}$ so that $\delta>0$ and if $0<|x-2|<\delta$ we must have $|x^2-4|<\varepsilon$.

Example 1.11 (Proof of Example 1.7)**.** Let $\varepsilon>0$ be given. Then there exists a prime number p such that $\dfrac{1}{p}<\varepsilon$. Let $q_1, q_2, \cdots, q_n$ be rational numbers in $\big(\dfrac{c}{2}, \dfrac{3c}{2}\big)$ satisfying

$$q_j = \frac{s}{r}\,, (r,s)=1, 1\leqslant r\leqslant p\,,$$

and define $\delta=\dfrac{1}{2}\min\Big(\big\{|c-q_1|, |c-q_2|, \cdots, |c-q_n|\big\}-\{0\}\Big)$. Then $\delta>0$. Suppose that x satisfies that $0<|x-c|<\delta$.

1. If $x\in\mathbb{Q}^\complement$, then $f(x)=0$ which shows that $|f(x)|<\varepsilon$.
2. If $x\in\mathbb{Q}$, then $x=\dfrac{s}{r}$ for some natural numbers r,s satisfying $(r,s)=1$. By the choice of δ, we find that $r>p$; thus
$$\big|f(x)\big| = \frac{1}{r} < \frac{1}{p} < \varepsilon\,.$$

In either case, $\big|f(x)\big|<\varepsilon$; thus we establish that

$$\big|f(x)-0\big|<\varepsilon \quad \text{whenever} \quad 0<|x-c|<\delta\,.$$

Therefore, $\lim\limits_{x\to c}f(x)=0$.

Proposition 1.12

Let f, g be functions defined on an open interval containing c (except possibly at c), and $f(x) = g(x)$ if $x \neq c$. If $\lim_{x\to c} g(x) = L$, then $\lim_{x\to c} f(x) = L$.

Proof. Let $\varepsilon > 0$ be given. Since $\lim_{x\to c} g(x) = L$, there exists $\delta > 0$ such that

$$|g(x) - L| < \varepsilon \quad \text{if} \quad 0 < |x - c| < \delta.$$

Since $f(x) = g(x)$ if $x \neq c$, we must have if $0 < |x - c| < \delta$,

$$|f(x) - L| = |g(x) - L| < \varepsilon\,.$$

□

Example 1.13. Let $f(x) = x + 1$ and $g(x) = \dfrac{x^2 - 1}{x - 1}$. Since $f(x) = g(x)$ if $x \neq 1$, the proposition above implies that

$$\lim_{x\to 1} g(x) = \lim_{x\to 1} f(x) = 2\,.$$

1.2 Properties of Limits

Theorem 1.14

Let b, c be real numbers, f, g be functions defined on an open interval containing c (except possibly at c) with $\lim_{x\to c} f(x) = L$ and $\lim_{x\to c} g(x) = K$.

1. $\lim_{x\to c} b = b$, $\lim_{x\to c} x = c$, $\lim_{x\to c} |x| = |c|$;
2. $\lim_{x\to c} \big[f(x) \pm g(x)\big] = L + K$;（和或差的極限等於極限的和或差）
3. $\lim_{x\to c} \big[f(x)g(x)\big] = LK$;（乘積的極限等於極限的乘積）
4. $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}$ if $K \neq 0$.（若分母極限不為零，則商的極限等於極限的商）

Proof. 1. Let $\varepsilon > 0$ be given.

(a) Define $\delta = 1$. Then $\delta > 0$ and if $0 < |x - c| < \delta$, we have $|b - b| = 0 < \varepsilon$.

(b) Define $\delta = \varepsilon$. Then $\delta > 0$ and if $0 < |x - c| < \delta$, we have $|x - c| < \delta = \varepsilon$.

(c) Define $\delta = \varepsilon$. Then $\delta > 0$ and if $0 < |x - c| < \delta$, by the triangle inequality we have

$$\big||x| - |c|\big| \leqslant |x - c| < \delta = \varepsilon\,.$$

2. Let $\varepsilon > 0$ be given. Since $\lim\limits_{x\to c} f(x) = L$ and $\lim\limits_{x\to c} g(x) = K$, there exist $\delta_1, \delta_2 > 0$ such that

$$\big|f(x) - L\big| < \frac{\varepsilon}{2} \quad \text{whenever} \quad 0 < |x - c| < \delta_1$$

and

$$\big|g(x) - K\big| < \frac{\varepsilon}{2} \quad \text{whenever} \quad 0 < |x - c| < \delta_2\,.$$

Define $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta > 0$ and if $0 < |x - c| < \delta$, we have

$$\big|f(x) + g(x) - (L + K)\big| \leqslant \big|f(x) - L\big| + \big|g(x) - K\big| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.$$

3. Let $\varepsilon > 0$ be given. Since $\lim\limits_{x\to c} f(x) = L$, there exist $\delta_1,\ \delta_2 > 0$ such that

$$\big|f(x) - L\big| < 1 \quad \text{whenever} \quad 0 < |x - c| < \delta_1$$

and

$$\big|f(x) - L\big| < \frac{\varepsilon}{2(|K| + 1)} \quad \text{whenever} \quad 0 < |x - c| < \delta_2\,.$$

Moreover, since $\lim\limits_{x\to c} g(x) = K$, there exists $\delta_3 > 0$ such that

$$\big|g(x) - K\big| < \frac{\varepsilon}{2(|L| + 1)} \quad \text{whenever} \quad 0 < |x - c| < \delta_3\,.$$

Define $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then $\delta > 0$ and if $0 < |x - c| < \delta$, we have

$$\begin{aligned}
\big|f(x)g(x) - LK\big| &= \big|f(x)g(x) - f(x)K + f(x)K - LK\big| \\
&\leqslant \big|f(x)\big|\big|g(x) - K\big| + |K|\big|f(x) - L\big| \\
&< (|L| + 1)\frac{\varepsilon}{2(|L| + 1)} + |K|\frac{\varepsilon}{2(|K| + 1)} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\,.
\end{aligned}$$

4. W.L.O.G. (Without loss of generality), we can assume that $K > 0$ for otherwise we have $\lim\limits_{x\to c}(-g)(x) = -K > 0$ and

$$\lim_{x\to c} \Big(\frac{f}{g}\Big)(x) = \lim_{x\to c} \Big(\frac{-f}{-g}\Big)(x) = \frac{\lim\limits_{x\to c}(-f)(x)}{-K} = \frac{-L}{-K} = \frac{L}{K}\,.$$

Let $\varepsilon > 0$ be given. Since $\lim_{x\to c} g(x) = K$, there exist $\delta_1, \delta_2 > 0$ such that

$$|g(x) - K| < \frac{K}{2} \quad \text{whenever} \quad 0 < |x - c| < \delta_1$$

and

$$|g(x) - K| < \frac{K^2\varepsilon}{4(|L| + 1)} \quad \text{whenever} \quad 0 < |x - c| < \delta_2 \,.$$

Moreover, since $\lim_{x\to c} f(x) = L$, there exists $\delta_3 > 0$ such that

$$|f(x) - L| < \frac{K\varepsilon}{4} \quad \text{whenever} \quad 0 < |x - c| < \delta_3 \,.$$

Define $\delta = \min\{\delta_1, \delta_2, \delta_3\}$. Then $\delta > 0$ and if $0 < |x - c| < \delta$, we have

$$\begin{aligned}
\Big|\frac{f(x)}{g(x)} - \frac{L}{K}\Big| &= \frac{|Kf(x) - Lg(x)|}{K|g(x)|} \leqslant \frac{1}{|g(x)|}\frac{|Kf(x) - KL| + |KL - Lg(x)|}{K} \\
&\leqslant \frac{2}{K}\Big(|f(x) - L| + \frac{|L|}{K}|g(x) - K|\Big) \\
&< \frac{2}{K}\Big(\frac{K\varepsilon}{4} + \frac{|L|}{K}\frac{K^2\varepsilon}{4(|L| + 1)}\Big) \leqslant \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \,,
\end{aligned}$$

where we have used $\dfrac{2}{K} \leqslant \dfrac{1}{|g(x)|}$ if $0 < |x-c| < \delta$ to conclude the inequality. Therefore, we conclude that $\lim_{x\to c} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}$ if $K > 0$. □

Example 1.15. Find $\lim_{x\to 3} x^2$. By 1 of Theorem 1.14 $\lim_{x\to 3} x = 3$; thus 3 of Theorem 1.14 implies that

$$\lim_{x\to 3} x^2 = \Big(\lim_{x\to 3} x\Big)\Big(\lim_{x\to 3} x\Big) = 9 \,.$$

The above equality further shows that

$$\lim_{x\to 3} x^3 = \Big(\lim_{x\to 3} x^2\Big)\Big(\lim_{x\to 3} x\Big) = 27 \,.$$

In particular, if n is a positive integer, then (by induction) $\lim_{x\to c} x^n = c^n$.

Corollary 1.16

Assume the assumptions in Theorem 1.14, and let n be a positive integer.

1. $\lim\limits_{x\to c}\left[f(x)^n\right] = L^n$.
2. If p is a polynomial function, then $\lim\limits_{x\to c} p(x) = p(c)$.
3. If r is a rational function given by $r(x) = \dfrac{p(x)}{q(x)}$ for some polynomials p and q, and $q(c) \neq 0$, then $\lim\limits_{x\to c} r(x) = r(c)$.

An illustration of why 2 in Corollary 1.16 is correct: Suppose that $p(x) = 3x^2 + 5x - 10$. Then applying 1-3 in Theorem 1.14, we obtain that

$$\begin{aligned}\lim_{x\to c} p(x) &= \lim_{x\to c}(3x^2+5x) - \lim_{x\to c}(10) = \lim_{x\to c}(3x^2+5x) - 10\\ &= \Big(\lim_{x\to c}(3)\Big)\Big(\lim_{x\to c} x^2\Big) + \Big(\lim_{x\to c}(5)\Big)\Big(\lim_{x\to c} x\Big) - 10\\ &= 3c^2 + 5c - 10 = p(c)\,.\end{aligned}$$

Theorem 1.17

If $c > 0$ and n is a positive integer, then $\lim\limits_{x\to c} x^{\frac{1}{n}} = c^{\frac{1}{n}}$.

Proof. Let $\varepsilon > 0$ be given. Define $\delta = \min\left\{\dfrac{c}{2}, \dfrac{nc^{\frac{n-1}{n}}\varepsilon}{2}\right\}$. Then $\delta > 0$ and if $0 < |x-c| < \delta$, we must have

$$x^{\frac{n-1}{n}} + x^{\frac{n-2}{n}}c^{\frac{1}{n}} + x^{\frac{n-3}{n}}c^{\frac{2}{n}} + \cdots + x^{\frac{1}{n}}c^{\frac{n-2}{n}} + c^{\frac{n-1}{n}} \geqslant \frac{n}{2}c^{\frac{n-1}{n}}\,.$$

Therefore, if $0 < |x-c| < \delta$,

$$\begin{aligned}\big|x^{\frac{1}{n}} - c^{\frac{1}{n}}\big| &= \left|\frac{x-c}{x^{\frac{n-1}{n}} + x^{\frac{n-2}{n}}c^{\frac{1}{n}} + x^{\frac{n-3}{n}}c^{\frac{2}{n}} + \cdots + x^{\frac{1}{n}}c^{\frac{n-2}{n}} + c^{\frac{n-1}{n}}}\right|\\ &\leqslant \frac{2}{n}c^{-\frac{n-1}{n}}|x-c| < \frac{2}{n}c^{-\frac{n-1}{n}}\delta \leqslant \frac{2}{n}c^{-\frac{n-1}{n}}\frac{nc^{\frac{n-1}{n}}\varepsilon}{2} = \varepsilon\end{aligned}$$

which implies that $\lim\limits_{x\to c} x^{\frac{1}{n}} = c^{\frac{1}{n}}$. ☐

Theorem 1.18

If f and g are functions (defined on open intervals) such that $\lim_{x\to c} g(x) = K$, $\lim_{x\to K} f(x) = L$ and $L = f(K)$, then

$$\lim_{x\to c}(f \circ g)(x) = L\,.$$

Proof. Let $\varepsilon > 0$ be given. Since $\lim_{x\to L} f(x) = L$, there exists $\delta_1 > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - K| < \delta_1\,.$$

Since $L = f(K)$, the statement above implies that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad |x - K| < \delta_1\,.$$

Fix such δ_1. Since $\lim_{x\to c} g(x) = K$, there exists $\delta > 0$ such that

$$|g(x) - K| < \delta_1 \quad \text{whenever} \quad 0 < |x - c| < \delta\,.$$

Therefore, if $0 < |x - c| < \delta$, $|(f \circ g)(x) - L| = |f(g(x)) - L| < \varepsilon$ which concludes the theorem. $\square$

Example 1.19. Find $\lim_{x\to 0} \frac{\sqrt{x+1}-1}{x}$.

Let $f(x) = \frac{\sqrt{x+1}-1}{x}$. If $x \neq 0$,

$$f(x) = \frac{(\sqrt{x+1}-1)(\sqrt{x+1}+1)}{x(\sqrt{x+1}+1)} = \frac{1}{\sqrt{x+1}+1} \equiv g(x)\,.$$

To see the limit of g, note that

$$\lim_{x\to 0} \sqrt{x+1} = 1 \qquad \text{(by Theorem 1.18)};$$

thus by Theorem 1.14 $\lim_{x\to 0} g(x) = \frac{1}{2}$.

Remark 1.20. In Theorem 1.18, the condition $L = f(K)$ is important, even though intuitively if $g(x) \to K$ as $x \to c$ and $f(x) \to L$ as $x \to K$ then $(f \circ g)(x)$ should approach L as x approaches c. A counter-example is given by the following two functions: f is the function given in Example 1.3 and g is a constant function with value 2. This example/ theorem demonstrates an important fact: intuition could be wrong! That is the reason why mathematicians develop the ε-δ language in order to explain ideas of limits rigorously.

Theorem 1.21: Squeeze Theorem（夾擠定理）

Let f, g, h be functions defined on an open interval containing c (except possibly at c), and $h(x) \leqslant f(x) \leqslant g(x)$ if $x \neq c$. If $\lim\limits_{x \to c} h(x) = \lim\limits_{x \to c} g(x) = L$, then $\lim\limits_{x \to c} f(x)$ exists and is equal to L.

Proof. Let $\varepsilon > 0$. Since $\lim\limits_{x \to c} h(x) = \lim\limits_{x \to c} g(x) = L$, there exist $\delta_1, \delta_2 > 0$ such that

$$|h(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta_1$$

and

$$|g(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta_2 \,.$$

Define $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta > 0$ and if $0 < |x - c| < \delta$,

$$L - \varepsilon < h(x) \leqslant f(x) \leqslant g(x) < L + \varepsilon$$

which implies that $|f(x) - L| < \varepsilon$ whenever $0 < |x - c| < \delta$. □

Example 1.22. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x & \text{if } x \text{ is rational}\,, \\ -x & \text{if } x \text{ is irrational}\,. \end{cases}$$

Then $\lim\limits_{x \to c} f(x)$ D.N.E. if $c \neq 0$ and $\lim\limits_{x \to 0} f(x) = 0$.

1. If $c \neq 0$, then as $x \neq c$ approaches c and $x \in \mathbb{Q}$, $f(x)$ approaches c, while as $x \neq c$ approaches c and $x \notin \mathbb{Q}$, $f(x)$ approaches $-c$. This implies that as x approaches c, $f(x)$ does not approaches a fixed number; thus $\lim\limits_{x \to c} f(x)$ D.N.E.
2. Note that $\big|f(x)\big| = |x|$; thus $-|x| \leqslant f(x) \leqslant |x|$ for all $x \in \mathbb{R}$. Since $\lim\limits_{x \to 0} |x| = 0$, the Squeeze theorem implies that $\lim\limits_{x \to 0} f(x) = 0$.

Example 1.23. In this example we consider the limit of the sine function at a real number c. Before proceeding, let us first establish a fundamental inequality

$$|\sin x| \leqslant |x| \qquad \text{for all real numbers } x \text{ (in radian unit).} \tag{1.2.1}$$

Recall (0.2.2) that

$$\sin x \leqslant x \leqslant \tan x \qquad \forall\, 0 \leqslant x \leqslant \frac{\pi}{2} \,. \tag{0.2.2}$$

To see (1.2.1), it suffices to consider the case when $x \notin \big[0, \frac{\pi}{2}\big]$. Nevertheless,

1. it trivially holds that $|\sin x| \leqslant x$ if $x \geqslant \dfrac{\pi}{2}$;
2. if $x < 0$, then $|\sin x| = |\sin(-x)| \leqslant |-x| = |x|$.

Having establish (1.2.1), now note the sum-to-product formula implies that

$$|\sin x - \sin c| = 2\Big|\sin\frac{x-c}{2}\cos\frac{x+c}{2}\Big| \leqslant 2\Big|\sin\frac{x-c}{2}\Big| \leqslant |x-c| \qquad \text{for all real number } x.$$

Therefore, $\sin c - |x-c| \leqslant \sin x \leqslant \sin c + |x-c|$ for all real number x, and the Squeeze Theorem then implies that $\lim\limits_{x\to c}\sin x = \sin c$ since $\lim\limits_{x\to c}|x-c| = 0$.

Similarly, using the sum-to-product formula

$$\cos x - \cos c = -2\sin\frac{x+c}{2}\sin\frac{x-c}{2}\,,$$

we can also conclude that $\lim\limits_{x\to c}\cos x = \cos c$. The detail is left as an exercise.

By Theorem 1.14, Example 1.23 shows the following

Theorem 1.24

Let c be a real number in the domain of the given trigonometric functions.

1. $\lim\limits_{x\to c}\sin x = \sin c$;
2. $\lim\limits_{x\to c}\cos x = \sin c$;
3. $\lim\limits_{x\to c}\tan x = \tan c$;
4. $\lim\limits_{x\to c}\cot x = \cot c$;
5. $\lim\limits_{x\to c}\sec x = \sec c$;
6. $\lim\limits_{x\to c}\csc x = \csc c$.

Example 1.25. In this example we compute $\lim\limits_{x\to 0} x\sin\dfrac{1}{x}$ if it exists. Note that if the limit exists, we cannot apply 3 of Theorem 1.14 to find the limit since $\lim\limits_{x\to 0}\sin\dfrac{1}{x}$ does not exist. On the other hand, since $\Big|x\sin\dfrac{1}{x}\Big| \leqslant |x|$ if $x \neq 0$, $-|x| \leqslant x\sin\dfrac{1}{x} \leqslant |x|$ if $x \neq 0$. By the fact that $\lim\limits_{x\to 0}|x| = \lim\limits_{x\to 0}(-|x|) = 0$, the Squeeze Theorem implies that $\lim\limits_{x\to 0} x\sin\dfrac{1}{x} = 0$.

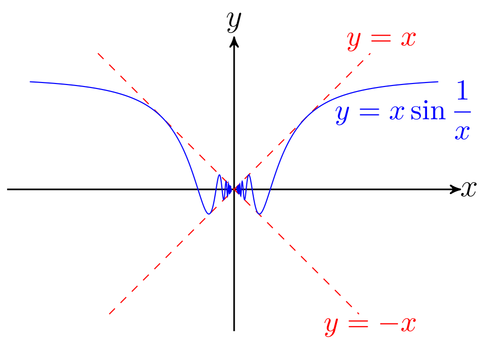

Figure 1.5: The graph of function $y = x\sin\dfrac{1}{x}$

1.2.1 One-sided limits and limits as $x \to \pm\infty$

Suppose that f is a function defined (only) on one side of a point c, it is also possible to consider the one-sided limit $\lim\limits_{x\to c^+} f(x)$ or $\lim\limits_{x\to c^-} f(x)$, where the notation $x \to c^+$ and $x \to c^-$ means that x is taken from the right-hand side and left-hand side of c, respectively, and becomes arbitrarily close to c. In other words, $\lim\limits_{x\to c^+} f(x)$ means the value to which $f(x)$ approaches as x approaches to c from the right, while $\lim\limits_{x\to c^-} f(x)$ means the value to which $f(x)$ approaches as x approaches to c from the left.

Definition 1.26: One-sided limits

Let f be a function defined on an interval with c as the left/right end-point (except possibly at c), and L be a real number. The statement

$$\lim_{x\to c^+} f(x) = L \Big/ \lim_{x\to c^-} f(x) = L\,,$$

read "the right/left(-hand) limit of f at c is L" or "the limit of f at c from the right/left is L", means that for each $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$\big|f(x) - L\big| < \varepsilon \quad \text{whenever} \quad 0 < x - c < \delta \big/ -\delta < x - c < 0\,.$$

Example 1.27. In this example we show that $\lim\limits_{x\to 0^+} x^{\frac{1}{n}} = 0$. Let $\varepsilon > 0$ be given. Define $\delta = \varepsilon^n$. Then $\delta > 0$ and if $0 < x < \delta$, we have

$$|x^{\frac{1}{n}} - 0| = x^{\frac{1}{n}} < \delta^{\frac{1}{n}} = \epsilon\,.$$

We note that Theorem 1.14, Theorem 1.17 and 1.21 are also valid when the limits are replaced by one-sided limits, and the precise statements are provided below.

Theorem 1.28

Let b, c be real numbers, f, g be functions with $\lim\limits_{x\to c^+} f(x) = L$ and $\lim\limits_{x\to c^+} g(x) = K$.

1. $\lim\limits_{x\to c^+} b = b$, $\lim\limits_{x\to c^+} x = c$, $\lim\limits_{x\to c^+} |x| = |c|$;
2. $\lim\limits_{x\to c^+} \big[f(x) \pm g(x)\big] = L + K$;
3. $\lim\limits_{x\to c^+} \big[f(x)g(x)\big] = LK$;
4. $\lim\limits_{x\to c^+} \dfrac{f(x)}{g(x)} = \dfrac{L}{K}$ if $K \neq 0$.

The conclusions above also hold for the case of left limits (that is, with $x \to c^+$ replaced by $x \to c^-$).

Theorem 1.29

If $c > 0$ and n is a positive integer, then $\lim\limits_{x\to c^+} x^{\frac{1}{n}} = c^{\frac{1}{n}}$ and $\lim\limits_{x\to c^-} x^{\frac{1}{n}} = c^{\frac{1}{n}}$

Theorem 1.30

If f and g are functions such that $\lim\limits_{x\to c^+} g(x) = K$, $\lim\limits_{x\to K} f(x) = L$ and $L = f(K)$, then

$$\lim_{x\to c^+} (f\circ g)(x) = L\,.$$

The conclusions above also hold for the case of left limits (that is, with $x \to c^+$ replaced by $x \to c^-$).

Remark 1.31. Theorem 1.30 is not true if one only has the one-sided limit $\lim\limits_{x\to K^+} f(x) = L$ instead of the full limit $\lim\limits_{x\to K} f(x) = L$. For example, consider $g(x) = -x$ and $f(x)$ be the function

$$f(x) = \begin{cases} 1 & \text{if } x \geqslant 0\,, \\ 0 & \text{if } x < 0\,. \end{cases}$$

Then $\lim\limits_{x\to 0^+} g(x) = 0$ and $\lim\limits_{x\to 0^+} f(x) = f(0)$; however,

$$(f\circ g)(x) = \begin{cases} 0 & \text{if } x > 0\,, \\ 1 & \text{if } x \leqslant 0\,, \end{cases}$$

which implies that $\lim\limits_{x\to 0^+} (f\circ g)(x) = 0 \neq f(0)$.

Theorem 1.32: Squeeze Theorem（夾擠定理）

1. Let f, g, h be functions defined on an interval with c as the left end-point (except possible at c), and $h(x) \leqslant f(x) \leqslant g(x)$ if $x > c$. If $\lim\limits_{x\to c^+} h(x) = \lim\limits_{x\to c^+} g(x) = L$, then $\lim\limits_{x\to c^+} f(x)$ exists and is equal to L.
2. Let f, g, h be functions defined on an interval with c as the right end-point (except possible at c), and $h(x) \leqslant f(x) \leqslant g(x)$ if $x < c$. If $\lim\limits_{x\to c^-} h(x) = \lim\limits_{x\to c^-} g(x) = L$, then $\lim\limits_{x\to c^-} f(x)$ exists and is equal to L.

The following theorem shows the relation between the limit and one-sided limits of functions.

Theorem 1.33

Let f be a function defined on an open interval containing c (except possibly at c). The limit $\lim\limits_{x\to c} f(x)$ exists if and only if $\lim\limits_{x\to c^+} f(x)$ and $\lim\limits_{x\to c^-} f(x)$ both exist and are identical. In either case,

$$\lim_{x\to c} f(x) = \lim_{x\to c^+} f(x) = \lim_{x\to c^-} f(x)\,.$$

Explanation on "A if and only if B" in Theorem 1.33: It should be clear that "A if B" means "A happens when B happens" (which is the same as "B implies A"). The statement "A only if B" means that "A happens only when B happens"; thus "A only if B" means that "A implies B".

Proof of Theorem 1.33. ($\Rightarrow$) - the "only if" part: Suppose that $\lim\limits_{x\to c} f(x) = L$, and let $\varepsilon > 0$ be given. Then there exists $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta\,.$$

Therefore, there exists $\delta > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < x - c < \delta\,;$$

thus $\lim\limits_{x\to c^+} f(x) = L$. Similarly, $\lim\limits_{x\to c^-} f(x) = L$.

($\Leftarrow$) - the "if" part: Suppose that $\lim\limits_{x\to c^+} f(x) = \lim\limits_{x\to c^-} f(x) = L$. Let $\varepsilon > 0$. Then there exist $\delta_1, \delta_2 > 0$ such that

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad 0 < x - c < \delta_1$$

and

$$|f(x) - L| < \varepsilon \quad \text{whenever} \quad -\delta_2 < x - c < 0\,.$$

Define $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta > 0$ and if $0 < |x - c| < \delta$, we must have $0 < x - c < \delta_1$ and $-\delta_2 < x - c < 0$; thus if $0 < |x - c| < \delta$, we must have $|f(x) - L| < \varepsilon$. $\square$

Example 1.34. In this example we compute a very important limit

$$\lim_{x\to 0} \frac{\sin x}{x} = 1\,. \tag{1.2.2}$$

To see this, we recall (0.2.2) that

$$\sin x \leqslant x \leqslant \tan x \quad \text{for all } 0 \leqslant x \leqslant \frac{\pi}{2}\,. \tag{0.2.2}$$

Now using (0.2.2), we find that

$$\cos x \leqslant \frac{\sin x}{x} \leqslant 1 \quad \text{for all } 0 < x < \frac{\pi}{2}\,.$$

The Squeeze Theorem (Theorem 1.32) then implies that $\lim\limits_{x\to 0^+} \frac{\sin x}{x} = 1$. On the other hand,

$$\lim_{x\to 0^-} \frac{\sin x}{x} = \lim_{x\to 0^-} \frac{\sin(-x)}{-x} = \lim_{x\to 0^+} \frac{\sin x}{x} = 1\,;$$

thus Theorem 1.33 implies that $\lim\limits_{x\to 0} \frac{\sin x}{x} = 1$.

Remark 1.35. The function $\frac{\sin x}{x}$ is the famous (unnormalized) sinc function; that is, $\mathrm{sinc}(x) = \frac{\sin x}{x}$ and $\mathrm{sinc}(0) = 1$. The example above shows that $\lim\limits_{x\to 0} \mathrm{sinc}(x) = \mathrm{sinc}(0)$.

Example 1.36. In this example we compute the limit $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2}$. By the half-angle formula, $1 - \cos x = 2\sin^2 \frac{x}{2}$; thus

$$\frac{1-\cos x}{x^2} = \frac{2\sin^2 \frac{x}{2}}{x^2} = \frac{1}{2}\frac{\sin^2 \frac{x}{2}}{\big(\frac{x}{2}\big)^2} = \frac{1}{2}\mathrm{sinc}^2\Big(\frac{x}{2}\Big)\,.$$

Therefore, Theorem 1.18 implies that $\lim\limits_{x\to 0} \frac{1-\cos x}{x^2} = \frac{1}{2}$.

An open interval in the real number system can be unbounded. When the open interval on which f is defined is not bounded from above (which means there is no real number which is larger than all the numbers in this interval), we can also consider the behavior of $f(x)$ as x becomes increasingly large and eventually outgrow all finite bounds.

Definition 1.37: Limits as $x \to \pm\infty$

Let f be a function defined on an infinite interval bounded from below/above, and L be a real number. The statement

$$\lim_{x\to\infty} f(x) = L \Big/ \lim_{x\to-\infty} f(x) = L\,,$$

read "the right/left(-hand) limit of f at c is L" or "the limit of f at c from the right/left is L", means that for each $\varepsilon > 0$ there exists a real number $M > 0$ such that

$$\big|f(x) - L\big| < \varepsilon \quad \text{whenever} \quad x > M \big/ x < -M\,.$$

Similar to the case of one-sided limit, Theorem 1.28, Theorem 1.30 and 1.32 are also valid when the notation $x \to c^\pm$ are replaced by $x \to \pm\infty$.

Example 1.38. In this example we show that $\lim\limits_{x\to\infty}\dfrac{1}{|x|}=0$ and $\lim\limits_{x\to-\infty}\dfrac{1}{|x|}=0$.

Let $\varepsilon>0$ be given. Define $M=\dfrac{1}{\varepsilon}$. Then if $x>M$ or $x<-M$, we must have $|x|>M$; thus if $x>M$ or $x<-M$,

$$\Big|\frac{1}{x}-0\Big|=\frac{1}{|x|}<\frac{1}{M}<\varepsilon\,.$$

Example 1.39. Recall that the sinc function is defined by

$$\operatorname{sinc}(x)=\begin{cases}\dfrac{\sin x}{x} & \text{if } x\neq 0\,,\\ 1 & \text{if } x=0\,.\end{cases}$$

Then $\Big|\dfrac{\sin x}{x}\Big|\leqslant\dfrac{1}{|x|}$ for all $x\neq 0$ and this provides the inequality $-\dfrac{1}{|x|}\leqslant\dfrac{\sin x}{x}\leqslant\dfrac{1}{|x|}$ for all $x\neq 0$. By the Squeeze Theorem and the previous example, we find that

$$\lim_{x\to\infty}\operatorname{sinc}(x)=\lim_{x\to-\infty}\operatorname{sinc}(x)=0\,.$$

Theorem 1.40

Let f be a function defined on an open interval, and $g(x)=f\big(\frac{1}{x}\big)$ if $x\neq 0$.

1. Suppose that the open interval is not bounded from above. Then $\lim\limits_{x\to\infty}f(x)$ exists if and only if $\lim\limits_{x\to 0^+}g(x)$ exists. In either case,
$$\lim_{x\to\infty}f(x)=\lim_{x\to 0^+}g(x)\,.$$
2. Suppose that the open interval is not bounded from below. Then $\lim\limits_{x\to-\infty}f(x)$ exists if and only if $\lim\limits_{y\to 0^-}g(x)$ exists. In either case,
$$\lim_{x\to-\infty}f(x)=\lim_{x\to 0^-}g(x)\,.$$

The theorem above should be very intuitive, and the proof is left as an exercise.

Example 1.41. Find the limit $\lim\limits_{x\to\infty}\dfrac{x+\sin x}{x+1}$.

By Theorem 1.40, we have

$$\begin{aligned}\lim_{x\to\infty}\frac{x+\sin x}{x+1}&=\lim_{x\to 0^+}\frac{\frac{1}{x}+\sin\frac{1}{x}}{\frac{1}{x}+1}=\lim_{x\to 0^+}\frac{1+x\sin\frac{1}{x}}{1+x}\\&=\lim_{x\to 0^+}\frac{1}{1+x}+\Big(\lim_{x\to 0^+}\frac{1}{x+1}\Big)\Big(\lim_{x\to 0^+}x\sin\frac{1}{x}\Big)=1+1\cdot 0=1\,.\end{aligned}$$

Here we note that in the process of computing the limit we have used results analogous to Theorem 1.28. We can also apply the Squeeze theorem to the inequality $\frac{x-1}{x+1} \leqslant \frac{x+\sin x}{x+1} \leqslant 1$ for all $x > 0$ and obtain the same limit.

Corollary 1.42

Let p and q be polynomial functions.

1. If the degree of p is smaller than the degree of q, then
$$\lim_{x\to\infty} \frac{p(x)}{q(x)} = \lim_{x\to-\infty} \frac{p(x)}{q(x)} = 0\,.$$
2. If the degree of p is the same as the degree of q, then
$$\lim_{x\to\infty} \frac{p(x)}{q(x)} = \lim_{x\to-\infty} \frac{p(x)}{q(x)} = \frac{\text{the leading coefficient of } p}{\text{the leading coefficient of } q}\,.$$

1.3 Continuity of Functions

Definition 1.43

Let f be a function defined on an interval I, and $c \in I$.

1. f is said to be right-continuous at c (or continuous from the right at c) if
$$\lim_{x\to c^+} f(x) = f(c)\,.$$
2. f is said to be left-continuous at c (or continuous from the left at c) if
$$\lim_{x\to c^-} f(x) = f(c)\,.$$
3. If c is the left end-point of I, f is said to be continuous at c if f is right-continuous at c.
4. If c is the right end-point of I, f is said to be continuous at c if f is left-continuous at c.
5. If c is an interior point of I; that is, c is neither the left end-point nor the right end-point of I, then f is said to be continuous at c if $\lim\limits_{x\to c} f(x) = f(c)$.

f is said to be discontinuous at c if f is not continuous at c, and in this case c is called a point of discontinuity (or simply a discontinuity) of f. f is said to be continuous (or a continuous function) on I if f is continuous at each point of I.

Example 1.44. Consider the the greatest integer function (also known as the Gauss function or the floor function) $[\![\cdot]\!] : \mathbb{R} \to \mathbb{R}$ defined by

$$[\![x]\!] = \text{the greatest integer which is not greater than } x.$$

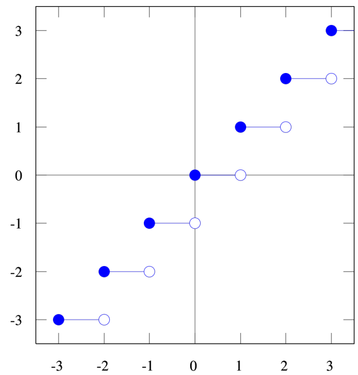

Figure 1.6: The greatest integer function $y = [\![x]\!]$

For example, $[\![2.5]\!] = 2$ and $[\![-2.5]\!] = -3$. If c is not an integer, $\lim\limits_{x\to c} [\![x]\!] = c$, while if c is an integer, we have

$$\lim_{x\to c^+} [\![x]\!] = c \qquad \text{and} \qquad \lim_{x\to c^-} [\![x]\!] = c - 1\,.$$

Let $f : [0,2] \to \mathbb{R}$ be given by $f(x) = [\![x]\!]$. Then the conclusion above shows that f is continuous at every non-integer number, while f is not continuous at 1 (since $\lim\limits_{x\to 1} f(x)$ does not exist) and 2 (since $\lim\limits_{x\to 2^-} f(x) \neq f(2)$). On the other hand, $\lim\limits_{x\to 0^+} f(x) = f(0)$, so f is continuous at 0.

Therefore, f is continuous at c if c is not an integer, and f is right-continuous at c if c is an integer.

Example 1.45. Let $f(x) = x^n$, where n is a positive integer. We have shown that

$$\lim_{x\to c} x^n = c^n$$

for all real numbers c; thus f is continuous on $\mathbb{R}$. In general, polynomial functions are continuous on $\mathbb{R}$ (because of Corollary 1.16).

Example 1.46. Let n be a positive integer, and $f : [0,\infty) \to \mathbb{R}$ be defined by $f(x) = x^{\frac{1}{n}}$. By Theorem 1.17 and Example 1.27,

$$\lim_{x\to c} x^{\frac{1}{n}} = c^{\frac{1}{n}} \quad \text{if } c > 0 \qquad \text{and} \qquad \lim_{x\to 0^+} x^{\frac{1}{n}} = 0\,;$$

thus f is continuous on $[0,\infty)$.

Example 1.47. Recall the Dirichlet function $f : \mathbb{R} \to \mathbb{R}$ in Example 1.6 given by

$$f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q}\,, \\ 1 & \text{if } x \notin \mathbb{Q}\,, \end{cases}$$

We have explained (but not proven) that the limit $\lim\limits_{x \to c} f(x)$ does not exist for all $c \in (0, \infty)$; thus f is discontinuous at all real numbers.

Example 1.48. Recall the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q}\,, \\ -x & \text{if } x \notin \mathbb{Q}\,. \end{cases}$$

in Example 1.22. We have shown that $\lim\limits_{x \to 0} f(x) = 0$; thus f is continuous at 0.

Example 1.49. Recall the function $f : (0, \infty) \to \mathbb{R}$ in Example 1.7 given by

$$f(x) = \begin{cases} \dfrac{1}{p} & \text{if } x = \dfrac{q}{p}, \text{ where } p, q \in \mathbb{N} \text{ and } (p, q) = 1\,, \\ 0 & \text{if } x \text{ is irrational}\,. \end{cases}$$

We have shown that $\lim\limits_{x \to c} f(x) = 0$ for all $c \in (0, \infty)$. Therefore, f is continuous at all irrational numbers but is discontinuous at all rational numbers.

Example 1.50. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous, and $f(x) = 2$ if $x \in \mathbb{Q}$. Then intuitively $f(x) = 2$ for all $x \in \mathbb{R}$. We now prove this using the definition of continuity.

Suppose the contrary that there exists $c \in \mathbb{R}$ such that $f(c) \neq 2$. Define $\varepsilon = \big|f(c) - 2\big|$. Then $\varepsilon > 0$. Since f is continuous at c, there exists $\delta > 0$ such that

$$\big|f(x) - f(c)\big| < \varepsilon \quad \text{whenever} \quad |x - c| < \delta\,.$$

Choose $x \in \mathbb{Q}$ such that $|x - c| < \delta$. Then the triangle inequality implies that

$$\varepsilon = \big|f(c) - 2\big| \leqslant \big|f(c) - f(x)\big| + \big|f(x) - 2\big| < \varepsilon$$

which is a contradiction.

Remark 1.51. Let I be an interval, $c \in I$, and $f : I \to \mathbb{R}$ be a function. The continuity of f at c is equivalent to that for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon \quad \text{whenever} \quad |x - c| < \delta \ \text{ and } \ x \in I\,.$$

To see this, we first consider the case that c is an interior point of I. Then by the definition, f is continuous at c if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon \quad \text{whenever} \quad 0 < |x - c| < \delta\,.$$

Since $|f(x) - f(c)| < \varepsilon$ automatically holds if $|x - c| = 0$, the statement above is equivalent to that

$$|f(x) - f(c)| < \varepsilon \quad \text{whenever} \quad |x - c| < \delta\,.$$

Now let us look at the case when c is the left end-point of I (so in this case $c \in I$). Then by definition, f is continuous at c if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|f(x) - f(c)| < \varepsilon \quad \text{whenever} \quad 0 < x - c < \delta\,.$$

Again $|f(x) - f(c)| < \varepsilon$ automatically holds if $x - c = 0$, the statement above is equivalent to that

$$|f(x) - f(c)| < \varepsilon \quad \text{whenever} \quad c \leqslant x < c + \delta\,.$$

Note that since c is the left end-point, the set $\{x \mid c \leqslant x < c+\delta\}$ is the same as $\{x \mid |x-c| < \delta, x \in I\}$; thus the statement above is equivalent to that

$$|f(x) - f(c)| < \varepsilon \quad \text{whenever} \quad |x - c| < \delta \ \text{ and } \ x \in I\,.$$

Similar argument can be applied to the case when c is the right end-point of I.

Remark 1.52. Discontinuities of functions can be classified into different categories: removable discontinuities and non-removable discontinuities. Let c be a discontinuity of a function f. Then either (1) $\lim\limits_{x\to c} f(x)$ exists but $\lim\limits_{x\to c} f(x) \neq f(c)$ or (2) $\lim\limits_{x\to c} f(x)$ does not exist. If it is the first case, then c is called a ***removable discontinuity*** and that means we can adjust/re-define the value of f at c to make it continuous at c. For the second case, no matter what $f(c)$ is, f cannot be continuous at c.

If $\lim\limits_{x\to c^+} f(x)$ and $\lim\limits_{x\to c^-} f(x)$ both exist but are not identical, c is also called a ***jump discontinuity***.

Proposition 1.53

Let f, g be defined on an interval I, $c \in I$, and f, g be continuous at c. Then

1. $f \pm g$ is continuous at c.
2. fg is continuous at c.
3. $\dfrac{f}{g}$ is continuous at c if $g(c) \neq 0$.

Corollary 1.54

Let f, g be continuous functions on an interval I. Then

1. $f \pm g$ is continuous on I.
2. fg is continuous on I
3. $\dfrac{f}{g}$ is continuous (on its domain).

Theorem 1.55

Let I, J be ~~open~~ intervals,, $g : I \to \mathbb{R}$, $f : J \to \mathbb{R}$ be functions, and J contains the range of g. If g is continuous at c, then $f \circ g$ is continuous at c.

Proof. Let $\varepsilon > 0$ be given. Since f is continuous at $g(c)$, there exists $\delta_1 > 0$ such that

$$\big|f(y) - f\big(g(c)\big)\big| < \varepsilon \qquad \text{whenever} \quad \big|y - g(c)\big| < \delta_1 \ \text{ and } \ y \in J\,.$$

For such a δ_1, by the continuity of g at c there exists $\delta > 0$ such that

$$\big|g(x) - g(c)\big| < \delta_1 \qquad \text{whenever} \quad |x - c| < \delta \ \text{ and } \ x \in I\,.$$

Suppose that $|x - c| < \delta$ and $x \in I$. Let $y = g(x)$. By the condition that J contains the range of g,

$$\big|y - g(c)\big| < \delta_1 \ \text{ and } \ y \in J\,.$$

Therefore, if $|x - c| < \delta$ and $x \in I$,

$$\big|f(g(x)) - f(g(c))\big| < \varepsilon$$

which shows the continuity of $f \circ g$ at c. □

Corollary 1.56

Let I, J be ~~open~~ intervals, and $g : I \to \mathbb{R}$, $f : J \to \mathbb{R}$ be continuous functions. If J contains the range of g, then $f \circ g$ is continuous on I.

Example 1.57. Let g be continuous on an interval I, and n be a positive integer. We show that g^n and $|g|^{\frac{1}{n}}$ are also continuous on I. Note that g^n is the function given by $g^n(x) = g(x)^n$ and $|g|^{\frac{1}{n}}$ is the function given by $|g|^{\frac{1}{n}} = |g(x)|^{\frac{1}{n}}$.

1. Let $f(x) = x^n$. Then Theorem 1.14 (or Corollary 1.16) implies that f is continuous on $\mathbb{R}$. Since $\mathbb{R}$ contains the range of g, by the corollary above we find tat $f \circ g\,(\equiv g^n)$ is continuous on I.

2. Let $h(x) = |x|$. Then Theorem 1.14 implies that h is continuous on $\mathbb{R}$. Since $\mathbb{R}$ contains the range of g, by the corollary above we find that $h \circ g\,(\equiv |g|)$ is continuous on I.

   Let $f(x) = x^{\frac{1}{n}}$. Then Theorem 1.17 and Example 1.27 imply that f is continuous on the non-negative real axis $[0, \infty)$. Since $[0, \infty)$ contains the range of $|g|$, the corollary above shows that $f \circ |g| (\equiv |g|^{\frac{1}{n}})$ is continuous on I.

Theorem 1.58: Intermediate Value Theorem - 中間值定理

If f is continuous on the closed interval $[a, b]$, $f(a) \neq f(b)$, and k is any number between $f(a)$ and $f(b)$, then there is at least one number c in $[a, b]$ such that $f(c) = k$.

Example 1.59 (Bisection method of finding zeros of continuous functions)**.** Let f be a function and $f(a)f(b) < 0$. Then the intermediate value theorem implies that there exists a zero c of f between a and b. How do we "find" (one of) this c? Consider the middle point $\dfrac{a+b}{2}$ of a and b. If $f\big(\dfrac{a+b}{2}\big) = 0$, then we find this zero, or otherwise we either have

$$f(a)f\big(\frac{a+b}{2}\big) < 0 \qquad \text{or} \qquad f(b)f\big(\frac{a+b}{2}\big) < 0$$

and only one of them can happen. In either case we can consider the middle point of the two points at which the value of f have different sign. Continuing this process, we can locate one zero as accurate as possible.

Example 1.60. Let $f : [0, 1] \to [0, 1]$ be a continuous function. In the following we prove that there exists $c \in [0, 1]$ such that $f(c) = c$. To see this, W.L.O.G. we assume that $f(0) \neq 0$ and $f(1) \neq 1$ for otherwise we find c (which is 0 or 1) such that $f(c) = c$.

Define $g(x) = f(x) - x$. Then g is continuous (by Proposition 1.53). Since $f : [0, 1] \to [0, 1]$, $f(0) \neq 0$ and $f(1) \neq 1$, we must have $g(0) > 0$ and $g(1) < 0$. By the intermediate value theorem, there exists $c \in (0, 1)$ such that $g(c) = 0$, and this implies that there exists $c \in (0, 1)$ such that $f(c) = c$. So either (1) $f(0) = 0$, (2) $f(1) = 1$, or (3) there ia $c \in (0, 1)$ such that $f(c) = c$.

1.4 Infinite Limits and Asymptotes

Definition 1.61

Let f be defined on an open interval containing c (except possible at c). The statement

$$\lim_{x\to c} f(x) = \infty\,,$$

read "$f(x)$ approaches infinity as x approaches c", means that for every $N > 0$ there exists $\delta > 0$ such that

$$f(x) > N \quad \text{whenever} \quad 0 < |x - c| < \delta\,.$$

The statement

$$\lim_{x\to c} f(x) = \infty\,,$$

read "$f(x)$ approaches minus infinity as x approaches c", means that for every $N > 0$ there exists $\delta > 0$ such that

$$f(x) < -N \quad \text{whenever} \quad 0 < |x - c| < \delta\,.$$

To define the infinite limit from the left/right, replace $0 < |x - c| < \delta$ by $c < x < c + \delta/c - \delta < x < c$. To define the infinite limit as $x \to \infty/x \to -\infty$, replace $0 < |x - c| < \delta$ by $x > \delta/x < -\delta$.

Note that the statement $\lim\limits_{x\to c} f(x) = \infty$ does **not** mean that the limit exists. It is a simple notation for saying that the value of f becomes unbounded as x approaches c and the limit fail to exist.

Example 1.62. $\lim\limits_{x\to 1} \dfrac{1}{(x-1)^2} = \infty$, $\lim\limits_{x\to 1^+} \dfrac{1}{x-1} = \infty$, and $\lim\limits_{x\to 1^-} \dfrac{1}{x-1} = -\infty$.

Example 1.63. Later we will talk about the exponential function in detail. In the mean time, assume that you know the graph of $y = 2^x$. Then $\lim\limits_{x\to\infty} 2^x = \infty$ and $\lim\limits_{x\to-\infty} 2^x = 0$.

• **Asymptotes**（漸近線）: If the distance between the graph of a function and some fixed straight line approaches zero as a point on the graph moves increasingly far from the origin, we say that the graph approaches the line asymptotically and that the line is an asymptote of the graph.

Definition 1.64: Vertical Asymptotes - 垂直漸近線

If f approaches infinity (or minus infinity) as x approaches c from the left or from the right, then the line $x = c$ is called a vertical asymptote of the graph of f.

Definition 1.65: Horizontal and Slant (Oblique) Asymptotes - 水平與斜漸近線

The straight line $y = mx + k$ is an asymptote of the graph of the function $y = f(x)$ if

$$\lim_{x\to\infty} \big[f(x) - mx - k\big] = 0 \qquad \text{or} \qquad \lim_{x\to-\infty} \big[f(x) - mx - k\big] = 0\,.$$

The straight line $y = mx + k$ is called a horizontal asymptote of the graph of f if $m = 0$, and is called a slant (oblique) asymptote of the graph of f if $m \neq 0$.

By the definition of horizontal asymptotes, it is clear that if $\lim\limits_{x\to\infty} f(x) = k$ or $\lim\limits_{x\to-\infty} f(x) = k$, then $y = k$ is a horizontal asymptote of the graph of f.

Example 1.66. Let $f(x) = \dfrac{x^2+3}{3x^2-4x+5}$. Then $\lim\limits_{x\to\infty} f(x) = \lim\limits_{x\to-\infty} f(x) = \dfrac{1}{3}$; thus $y = \dfrac{1}{3}$ is a horizontal asymptote of the graph of f.

Example 1.67. Let $f(x) = \dfrac{x^3+3}{3x^2-4x+5}$. Then $\lim\limits_{x\to\infty} f(x) = \infty$ and $\lim\limits_{x\to-\infty} f(x) = -\infty$; thus the graph of f has no horizontal asymptote. However,

$$\lim_{x\to\infty} \Big[f(x) - \frac{x}{3}\Big] = \lim_{x\to\infty} \Big[\frac{3x^3+9}{3(3x^2-4x+5)} - \frac{x(3x^2-4x+5)}{3(3x^2-4x+5)}\Big] = \lim_{x\to\infty} \frac{4x^2-5x+9}{3(3x^2-4x+5)} = \frac{4}{9}\,;$$

thus $\lim\limits_{x\to\infty} \Big[f(x) - \dfrac{x}{3} - \dfrac{4}{9}\Big] = 0$. Therefore, $y = \dfrac{x}{3} + \dfrac{4}{9}$ is a slant asymptote of the graph of f.

Theorem 1.68

Let f and g be continuous on an open interval containing c. If $f(c) \neq 0$, $g(c) = 0$, and there exists an open interval containing c such that $g(x) \neq 0$ for all $x \neq c$ in the interval, then the graph of the function $h(x) = \dfrac{f(x)}{g(x)}$ has a vertical asymptote at $x = c$.

Example 1.69. Let $f(x) = \tan x$. Note that $\tan x = \dfrac{\sin x}{\cos x}$. For $n \in \mathbb{Z}$, $\sin\big(n\pi + \dfrac{\pi}{2}\big) \neq 0$ and $\cos\big(n\pi + \dfrac{\pi}{2}\big) = 0$. Moreover, $\cos x \neq 0$ for every x in the open interval $\big(n\pi + \dfrac{\pi}{4}, n\pi + \dfrac{3\pi}{4}\big)$ except $n\pi + \dfrac{\pi}{2}$. Therefore, by the theorem above we find that $x = n\pi + \dfrac{\pi}{2}$ is a vertical asymptote of the graph of the tangent function for all $n \in \mathbb{Z}$.

Theorem 1.70

If $y = mx + k$ is a slant asymptote of the graph of the function $y = f(x)$, then

$$m = \lim_{x\to\infty} \frac{f(x)}{x} \qquad \text{or} \qquad m = \lim_{x\to-\infty} \frac{f(x)}{x}$$

and

$$k = \lim_{x\to\infty} \big[f(x) - mx\big] \qquad \text{of} \qquad k = \lim_{x\to-\infty} \big[f(x) - mx\big]\,.$$

Proof. It suffices to shows that $m = \lim\limits_{x\to\infty} \dfrac{f(x)}{x}$ or $m = \lim\limits_{x\to-\infty} \dfrac{f(x)}{x}$. W.L.O.G., we assume that $\lim\limits_{x\to\infty} \big[f(x) - mx - k\big] = 0$. Then

$$\lim_{x\to\infty} \frac{f(x) - mx - k}{x} = 0\,.$$

On the other hand, $\lim\limits_{x\to\infty} \dfrac{mx + k}{x} = m$. By the fact that $\dfrac{f(x)}{x} = \dfrac{f(x) - mx - k}{x} + \dfrac{mx + k}{x}$, we find that $\lim\limits_{x\to\infty} \dfrac{f(x)}{x}$ exists and

$$\lim_{x\to\infty} \frac{f(x)}{x} = \lim_{x\to\infty} \Big[\frac{f(x) - mx - k}{x}\Big] + \lim_{x\to\infty} \frac{mx + k}{x} = m\,. \qquad \square$$

Example 1.71. In this example, we find all asymptotes of the graph of the function

$$f(x) = \frac{3x^3(x - \sqrt[3]{x^3 - x^2 + x})}{x^2 - 1}\,.$$

Since the denominator vanishes at $x = \pm 1$, there are two possible vertical asymptotes $x = 1$ or $x = -1$. Since the denominator also vanishes at $x = 1$, we need to check further the behavior of $f(x)$ as x approaches 1. Note that for $x \neq \pm 1$,

$$\frac{x - \sqrt[3]{x^3 - x^2 + x}}{x^2 - 1} = \frac{x}{(x + 1)\big[x^2 + x\sqrt[3]{x^3 - x^2 + x} + (x^3 - x^2 + x)^{\frac{2}{3}}\big]}\,;$$

thus for $x \neq \pm 1$,

$$f(x) = \frac{3x^4}{(x + 1)\big[x^2 + x\sqrt[3]{x^3 - x^2 + x} + (x^3 - x^2 + x)^{\frac{2}{3}}\big]}\,.$$

Therefore, $\lim\limits_{x\to 1} f(x) = 0$ exists which shows that $x = 1$ is not a vertical asymptote of the graph of f. On the other hand,

$$\lim_{x\to-1^+} f(x) = \infty \qquad \text{and} \qquad \lim_{x\to-1^-} f(x) = -\infty\,,$$

we find that $x=-1$ is the only vertical asymptote of the graph of f.

For slant or horizontal asymptotes, we note that for $x \neq \pm 1, 0$,

$$\frac{f(x)}{x} = \frac{3}{\left(1+\frac{1}{x}\right)\left[1+\left(1-\frac{1}{x}+\frac{1}{x^2}\right)^{\frac{1}{3}}+\left(1-\frac{1}{x}+\frac{1}{x^2}\right)^{\frac{2}{3}}\right]}\,. \tag{1.4.1}$$

Since $\lim\limits_{x\to\pm\infty}\frac{1}{x}=0$, we find that $\lim\limits_{x\to\infty}\frac{f(x)}{x}=1$ and $\lim\limits_{x\to-\infty}\frac{f(x)}{x}=1$. It remains to find the limit $\lim\limits_{x\to\infty}\big[f(x)-x\big]$ and $\lim\limits_{x\to-\infty}\big[f(x)-x\big]$. Using (1.4.1),

$$f(x)-x = \frac{3x-(x+1)\left[1+\left(1-\frac{1}{x}+\frac{1}{x^2}\right)^{\frac{1}{3}}+\left(1-\frac{1}{x}+\frac{1}{x^2}\right)^{\frac{2}{3}}\right]}{\left(1+\frac{1}{x}\right)\left[1+\left(1-\frac{1}{x}+\frac{1}{x^2}\right)^{\frac{1}{3}}+\left(1-\frac{1}{x}+\frac{1}{x^2}\right)^{\frac{2}{3}}\right]}\,.$$

Noting that the denominator approaches 3 as x approaches $\pm\infty$, we only focus on the limit of the numerator. Since

$$\begin{aligned}
&3x-(x+1)\Big[1+\big(1-\frac{1}{x}+\frac{1}{x^2}\big)^{\frac{1}{3}}+\big(1-\frac{1}{x}+\frac{1}{x^2}\big)^{\frac{2}{3}}\Big]\\
&\quad= 3x-(x+1)\Big[3+\Big(\big(1-\frac{1}{x}+\frac{1}{x^2}\big)^{\frac{1}{3}}-1\Big)+\Big(\big(1-\frac{1}{x}+\frac{1}{x^2}\big)^{\frac{2}{3}}-1\Big)\Big]\\
&\quad= -3-\Big[\big(1-\frac{1}{x}+\frac{1}{x^2}\big)^{\frac{1}{3}}-1\Big]\Big[\big(1-\frac{1}{x}+\frac{1}{x^2}\big)^{\frac{1}{3}}+2\Big]\\
&\qquad -x\Big[\big(1-\frac{1}{x}+\frac{1}{x^2}\big)^{\frac{1}{3}}-1\Big]\Big[\big(1-\frac{1}{x}+\frac{1}{x^2}\big)^{\frac{1}{3}}+2\Big],
\end{aligned}$$

to find the limit of the numerator as $x\to\pm\infty$ it suffices to find the limit

$$\lim_{x\to\infty} x\Big[\big(1-\frac{1}{x}+\frac{1}{x^2}\big)^{\frac{1}{3}}-1\Big] \qquad \text{and} \qquad \lim_{x\to-\infty} x\Big[\big(1-\frac{1}{x}+\frac{1}{x^2}\big)^{\frac{1}{3}}-1\Big]\,.$$

Now, by Theorem 1.40,

$$\begin{aligned}
\lim_{x\to\infty} x\Big[\big(1-\frac{1}{x}+\frac{1}{x^2}\big)^{\frac{1}{3}}-1\Big] &= \lim_{x\to 0^+}\frac{\big(1-x+x^2\big)^{\frac{1}{3}}-1}{x}\\
&= \lim_{x\to 0^+}\frac{x-1}{\big(1-x+x^2\big)^{\frac{2}{3}}+\big(1-x+x^2\big)^{\frac{1}{3}}+1} = -\frac{1}{3}
\end{aligned}$$

and similarly, $\lim\limits_{x\to-\infty} x\Big[\big(1-\frac{1}{x}+\frac{1}{x^2}\big)^{\frac{1}{3}}-1\Big] = -\frac{1}{3}$. Therefore,

$$\lim_{x\to\pm\infty}\Big[3x-(x+1)\big[1-\frac{1}{x}+\frac{1}{x^2}\big)^{\frac{1}{3}}+\big(1-\frac{1}{x}+\frac{1}{x^2}\big)^{\frac{2}{3}}\big]\Big] = -3+\frac{1}{3}\cdot 3 = -2\,;$$

thus $\lim\limits_{x\to\pm\infty}\big[f(x)-x\big] = -\frac{2}{3}$ which implies that $y = x-\frac{2}{3}$ is the only slant asymptote of the graph of f.

Chapter 2

Differentiation

2.1 The Derivatives of Functions

Definition 2.1

Let f be a function defined on an open interval containing c. If the limit $\lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = m$ exists, then the line passing through $\big(c, f(c)\big)$ with slope m is the tangent line to the graph of f at point $\big((c, f(c)\big)$.

Definition 2.2

Let f be a function defined on an open interval I containing c. f is said to be differentiable at c if the limit

$$\lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

exists. If the limit above exists, the limit is denoted by $f'(c)$ and called the derivative of f at c. When the derivative of f at each point of I exists, f is said to be differentiable on I and the derivative of f is a function denoted by f'.

• **Notation**: The prime notation $'$ is associated with a function (of one variable) and is used to denote the derivative of that function. For a given function f defined on an open interval I and x being the name of the variable, the limit operation

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

is denoted by $\dfrac{d}{dx}f(x)$ $\big($or $\dfrac{df(x)}{dx}$ or even $\dfrac{dy}{dx}$ if $y = f(x)\big)$, and the limit

$$\lim_{\Delta x \to 0} \frac{f(c+\Delta) - f(c)}{\Delta x}$$

is denoted by $\dfrac{d}{dx}\Big|_{x=c} f(x)$ but not $\dfrac{d}{dx}f(c)$ $\big(\frac{d}{dx}f(c)$ is in fact $0\big)$. The operator $\dfrac{d}{dx}$ is a differential operator called the differentiation and is applied to functions of variable x. However, for historical (and convenient) reason, $\dfrac{d}{dx}f(x)$ is sometimes denoted by $(f(x))'$ (so that $'$ is treated as the differential operator $\dfrac{d}{dx}$) and f' is sometimes denoted by $\dfrac{df}{dx}$ (so that f is always treated as a function of variable x).

Remark 2.3. Letting $x = c + \Delta x$ in the definition of the derivatives, then

$$f'(c) = \lim_{x \to c} \frac{f(x) - f(c)}{x - c}$$

if the limit exists.

Example 2.4. Let f be a constant function. Then f' is the zero function.

Example 2.5. Let $f(x) = x^n$, where n is a positive integer. Then

$$f(x + \Delta x) = x^n + C^n_1 x^{n-1}\Delta x + C^n_2 x^{n-2}(\Delta x)^2 + \cdots + C^n_{n-1} x(\Delta x)^{n-1} + (\Delta x)^n\,;$$

thus if $\Delta x \neq 0$,

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = nx^{n-1} + C^n_2 x^{n-2}\Delta x + \cdots + C^n_{n-1} x(\Delta x)^{n-2} + (\Delta x)^{n-1}\,.$$

The limit on the right-hand side is clearly nx^{n-1}, so we establish that

$$\frac{d}{dx}x^n = nx^{n-1}\,.$$

Example 2.6. Now suppose that $f(x) = x^{-n}$, where n is a positive integer. Then if $x + \Delta x \neq 0$,

$$f(x+\Delta x) = \frac{1}{x^n + C^n_1 x^{n-1}\Delta x + C^n_2 x^{n-2}(\Delta x)^2 + \cdots + C^n_{n-1} x(\Delta x)^{n-1} + (\Delta x)^n}\,;$$

thus if $x \neq 0$, $\Delta x \neq 0$, and $x + \Delta x \neq 0$ (which can be achieved if $|\Delta x| \ll 1$),

$$\frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{-\big[C^n_1 x^{n-1} + C^n_2 x^{n-2}\Delta x + \cdots + C^n_{n-1} x(\Delta x)^{n-2} + (\Delta x)^{n-1}\big]}{x^n\big[x^n + C^n_1 x^{n-1}\Delta x + C^n_2 x^{n-2}(\Delta x)^2 + \cdots + C^n_{n-1} x(\Delta x)^{n-1} + (\Delta x)^n\big]}\,.$$

Therefore, if $x \neq 0$,

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \frac{-nx^{n-1}}{x^{2n}} = -nx^{-n-1}$$

which shows $\dfrac{d}{dx} x^{-n} = -nx^{-n-1}$.

Combining the previous three examples, we conclude that

$$\frac{d}{dx} x^n = \begin{cases} nx^{n-1} & \forall\, x \in \mathbb{R} \quad \text{if } n \in \mathbb{N} \cup \{0\}\,, \\ nx^{n-1} & \forall\, x \neq 0 \quad \text{if } n \in \mathbb{Z} \text{ and } n < 0\,. \end{cases} \tag{2.1.1}$$

Combining Example 2.4-2.6, we conclude that

$$\frac{d}{dx} x^n = \begin{cases} nx^{n-1} & \forall\, x \in \mathbb{R} \quad \text{if } n \in \mathbb{N} \cup \{0\}\,, \\ nx^{n-1} & \forall\, x \neq 0 \quad \text{if } n \in \mathbb{Z} \text{ and } n < 0\,. \end{cases} \tag{2.1.2}$$

我們注意到當 n 是負整數時，在計算 $\dfrac{d}{dx}\Big|_{x=c} x^n$ 時，已經必須先假設 $c \neq 0$ 才能計算導數，並非最後算出來 $\dfrac{d}{dx}\Big|_{x=c} x^n = nc^{n-1}$ 時發現 c 不可為零所以不能代入。這是一個非常重要的觀念！不能搞錯順序！

Example 2.7. Let $f(x) = \sin x$. By the sum and difference formula,

$$\begin{aligned} f(x+\Delta x) - f(x) &= \sin(x+\Delta x) - \sin x = \sin x \cos \Delta x + \sin \Delta x \cos x - \sin x \\ &= \sin x(\cos \Delta x - 1) + \sin \Delta x \cos x\,; \end{aligned}$$

thus by the fact that $\displaystyle\lim_{x \to 0} \frac{\sin x}{x} = 1$ and $\displaystyle\lim_{x \to 0} \frac{\cos x - 1}{x} = 0$, we find that

$$\lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \Big[\sin x \frac{\cos \Delta x - 1}{\Delta x} + \frac{\sin \Delta x}{\Delta x} \cos x\Big] = \cos x\,. \tag{2.1.3}$$

In other words, the derivative of the sine function is cosine.

On the other hand, let $g(x) = \cos x$. Then $g(x) = -f\big(x - \frac{\pi}{2}\big)$. Then if $\Delta x \neq 0$,

$$\frac{g(x+\Delta x) - g(x)}{\Delta x} = -\frac{f\big(x - \frac{\pi}{2} + \Delta x\big) - f\big(x - \frac{\pi}{2}\big)}{\Delta x}\,;$$

thus

$$\lim_{\Delta x \to 0} \frac{g(x+\Delta x) - g(x)}{\Delta x} = -\cos\big(x - \frac{\pi}{2}\big) = -\sin x\,.$$

In other words, the derivative of the cosine function is minus sine. To summarize,

$$\frac{d}{dx} \sin x = \cos x \qquad \text{and} \qquad \frac{d}{dx} \cos x = -\sin x\,. \tag{2.1.4}$$

Example 2.8. Consider the function $g : \mathbb{R} \to \mathbb{R}$ defined by

$$g(x) = \begin{cases} x^2 & \text{if } x \text{ is rational}\,, \\ -x^2 & \text{if } x \text{ is irrational}\,. \end{cases}$$

Then $g(x) = xf(x)$, where f is given in Example 1.22. By the fact that $\lim\limits_{x\to 0} f(x) = 0$,

$$\lim_{\Delta x\to 0} \frac{g(\Delta x) - g(0)}{\Delta x} = \lim_{\Delta x\to 0} f(\Delta x) = 0\,.$$

In other words, g is differentiable at 0. Moreover, similar argument used to explain that the function f in Example 1.22 is only continuous at 0 can be used to show that the function g is only continuous at 0. Therefore, we obtain a function which is differentiable at one point but discontinuous elsewhere.

Remark 2.9. If f is a function defined on a interval I, and c is one of the end-point. Then it is possible to define the one-sided derivative. For example, if c is the left end-point of I, then we can consider the limit

$$\lim_{\Delta x\to 0^+} \frac{f(c+\Delta x) - f(c)}{\Delta x} = \lim_{x\to c^+} \frac{f(x) - f(c)}{x - c}$$

if it exists. The limit above, if exists, is called the derivatives of f at c from the right.

Theorem 2.10: 可微必連續

Let f be a function defined on an open interval I, and $c \in I$. If f is differentiable at c, then f is continuous at c.

Proof. If $x \neq c$, $f(x) - f(c) = \dfrac{f(x) - f(c)}{x - c}(x - c)$. Since the limit $\lim\limits_{x\to c} \dfrac{f(x) - f(c)}{x - c}$ exists and $\lim\limits_{x\to c}(x - c) = 0$, by Theorem 1.14 we conclude that

$$\lim_{x\to c} \big[f(x) - f(c)\big] = \Big(\lim_{x\to c} \frac{f(x) - f(c)}{x - c}\Big)\Big(\lim_{x\to c}(x - c)\Big) = 0\,.$$

Therefore, $\lim\limits_{x\to c} f(x) = f(c)$ which shows that f is continuous at c. $\square$

Remark 2.11. When f is continuous on an open interval I, f is **not** necessary differentiable on I. For example, consider $f(x) = |x|$. Then Theorem 1.14 implies that f is continuous on I, but $\lim\limits_{\Delta x\to 0} \dfrac{f(\Delta x) - f(0)}{\Delta x} = \lim\limits_{\Delta x\to 0} \dfrac{|\Delta x|}{\Delta x}$ D.N.E.

2.2 Rules of Differentiation

Theorem 2.12

We have the following differentiation rules:

1. If k is a constant, then $\frac{d}{dx}k = 0$.

2. If n is a non-zero integer, then $\frac{d}{dx}x^n = nx^{n-1}$ (whenever x^{n-1} makes sense).

3. $\frac{d}{dx}\sin x = \cos x$, $\frac{d}{dx}\cos x = -\sin x$.

4. If k is a constant and $f : (a, b) \to \mathbb{R}$ is differentiable at $c \in (a, b)$, then kf is differentiable at c and
$$\frac{d}{dx}\Big|_{x=c}\big[kf(x)\big] = kf'(c)\,.$$

5. If $f, g : (a, b) \to \mathbb{R}$ are differentiable at $c \in (a, b)$, then $f \pm g$ is differentiable at c and
$$\frac{d}{dx}\Big|_{x=c}\big[f(x) \pm g(x)\big] = f'(c) \pm g'(c)\,.$$

Proof of 5. Let $h(x) = f(x) + g(x)$. Then if $\Delta x \neq 0$,

$$\frac{h(c + \Delta x) - h(c)}{\Delta x} = \frac{f(c + \Delta x) - f(c)}{\Delta x} + \frac{g(c + \Delta x) - g(c)}{\Delta x}\,.$$

Since f, g are differentiable at c,

$$\lim_{\Delta x \to 0} \frac{f(c + \Delta x) - f(c)}{\Delta x} = f'(c) \qquad \text{and} \qquad \lim_{\Delta x \to 0} \frac{g(c + \Delta x) - g(c)}{\Delta x}$$

exist. Therefore, by Theorem 1.14,

$$h'(c) = f'(c) + g'(c)\,.$$

The conclusion for the difference can be proved in the same way. □

Example 2.13. Let $f(x) = 3x^2 - 5x + 7$. Then

$$\begin{aligned}\frac{d}{dx}f(x) &= \frac{d}{dx}(3x^2 - 5x) + \frac{d}{dx}7 = \frac{d}{dx}(3x^2) - \frac{d}{dx}(5x)\\ &= 3\frac{d}{dx}x^2 - 5\frac{d}{dx}x = 3 \cdot (2x) - 5 = 6x - 5\,.\end{aligned}$$

In general, for a polynomial function

$$p(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0 \equiv \sum_{k=0}^{n} a_k x^k \,,$$

where $a_0, a_1, \cdots, a_n \in \mathbb{R}$, by induction we can show that

$$\frac{d}{dx} p(x) = n a_n x^{n-1} + (n-1) a_{n-1} x^{n-2} + \cdots + a_1 = \sum_{k=1}^{n} k a_k x^{k-1} \,.$$

Theorem 2.14: Product Rule

Let $f, g : (a, b) \to \mathbb{R}$ be real-valued functions, and $c \in (a, b)$. If f and g are differentiable at c, then fg is differentiable at c and

$$\left.\frac{d}{dx}\right|_{x=c} (fg)(x) = f'(c)g(c) + f(c)g'(c) \,.$$

Proof. Let $h(x) = f(x)g(x)$. Then

$$\begin{aligned} h(c+\Delta x) - h(c) &= f(c+\Delta x)g(c+\Delta x) - f(c)g(c) \\ &= f(c+\Delta x)g(c+\Delta x) - f(c)g(c+\Delta x) + f(c)g(c+\Delta x) - f(c)g(c) \\ &= \big[f(c+\Delta x) - f(c)\big] g(c+\Delta x) + f(c)\big[g(c+\Delta x) - g(c)\big] \,. \end{aligned}$$

Therefore, if $\Delta x \neq 0$,

$$\frac{h(c+\Delta x) - h(c)}{\Delta x} = \frac{f(c+\Delta x) - f(c)}{\Delta x} g(c+\Delta x) + f(c) \frac{g(c+\Delta x) - g(c)}{\Delta x} \,.$$

Since f, g are differentiable at c,

$$\lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = f'(c) \,, \lim_{\Delta x \to 0} \frac{g(c+\Delta x) - g(c)}{\Delta x} \,, \text{ and } \lim_{\Delta x \to 0} g(c+\Delta x) = g(c)$$

exist. By Theorem 1.14,

$$h'(c) = f'(c)g(c) + f(c)g'(c)$$

which concludes the product rule. □

Example 2.15. Let $f(x) = x^3 \sin x$. Then the product rule implies that

$$f'(x) = 3x^2 \sin x + x^3 \cos x \,.$$

Theorem 2.16: Quotient Rule

Let $f, g : (a,b) \to \mathbb{R}$ be real-valued functions, and $c \in (a,b)$. If f and g are differentiable at c and $g(c) \neq 0$, then $\frac{f}{g}$ is differentiable at c and

$$\frac{d}{dx}\Big|_{x=c} \frac{f}{g}(x) = \frac{f'(c)g(c) - f(c)g'(c)}{g(c)^2}\,.$$

Proof. Let $h(x) = \dfrac{f(x)}{g(x)}$. Then

$$\begin{aligned} h(c+\Delta x) - h(c) &= \frac{f(c+\Delta x)}{g(c+\Delta x)} - \frac{f(c)}{g(c)} = \frac{f(c+\Delta x)g(c) - f(c)g(c+\Delta x)}{g(c)g(c+\Delta x)} \\ &= \frac{f(c+\Delta x)g(c) - f(c)g(c) + f(c)g(c) - f(c)g(c+\Delta x)}{g(c)g(c+\Delta x)} \\ &= \frac{\big[f(c+\Delta x) - f(c)\big]g(c) - f(c)\big[g(c+\Delta x) - g(c)\big]}{g(c)g(c+\Delta x)}\,. \end{aligned}$$

Therefore, if $\Delta x \neq 0$,

$$\frac{h(c+\Delta x) - h(c)}{\Delta x} = \frac{1}{g(c)g(c+\Delta x)}\Big[\frac{f(c+\Delta x) - f(c)}{\Delta x} g(c) - f(c)\frac{g(c+\Delta x) - g(c)}{\Delta x}\Big]\,.$$

Since f, g are differentiable at c,

$$\lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x} = f'(c)\,, \lim_{\Delta x \to 0} \frac{g(c+\Delta x) - g(c)}{\Delta x}\,, \text{ and } \lim_{\Delta x \to 0} g(c+\Delta x) = g(c)$$

exist. By Theorem 1.14,

$$h'(c) = \frac{1}{g(c)^2}\Big[f'(c)g(c) - f(c)g'(c)\Big]$$

which concludes the quotient rule. □

Remark 2.17. Suppose that in addition to the assumption in Theorem 2.16 one has already known that $h = f/g$ is differentiable at c, then applying the product rule to $f = gh$ one finds that

$$f'(c) = g'(c)h(c) + g(c)h'(c) = g'(c)\frac{f(c)}{g(c)} + g(c)h'(c)$$

which, after rearranging terms, shows the quotient rule. The proof of Theorem 2.16 indeed is based on the fact that we do not know the differentiability of h at c yet.

Example 2.18. Let n be a positive integer and $f(x) = x^{-n}$. We have shown by definition that $f'(x) = -nx^{-n-1}$ if $x \neq 0$. Now we use Theorem 2.16 to compute the derivative of f: if $x \neq 0$,

$$\frac{d}{dx}x^{-n} = \frac{d}{dx}\frac{1}{x^n} = -\frac{\frac{d}{dx}x^n}{x^{2n}} = -\frac{nx^{n-1}}{x^{2n}} = -nx^{-n-1}\,.$$

Example 2.19. Since $\tan x = \dfrac{\sin x}{\cos x}$, by Theorem 2.16 we have

$$\frac{d}{dx}\tan x = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x\,.$$

Similarly, we also have

$$\begin{aligned}
\frac{d}{dx}\cot x &= \frac{-\sin^2 x - \cos^2 x}{\sin^2 x} = -\csc^2 x\,,\\
\frac{d}{dx}\sec x &= -\frac{-\sin x}{\cos^2 x} = \sec x \tan x\,,\\
\frac{d}{dx}\csc x &= -\frac{\cos x}{\sin^2 x} = -\csc x \cot x\,.
\end{aligned}$$

We note that without using the quotient rule, the derivative of the tangent function can be found using the sum-and-difference formula

$$\tan(x-y) = \frac{\tan x - \tan y}{1 + \tan x \tan y}\,. \tag{2.2.1}$$

Using (2.2.1), we find that

$$\tan(x+\Delta x) - \tan x = \tan \Delta x \big[1 + \tan(x+\Delta x)\tan x\big]\,;$$

thus if $\Delta x \neq 0$,

$$\frac{\tan(x+\Delta x) - \tan x}{\Delta x} = \frac{\sin \Delta x}{\Delta x} \cdot \frac{1 + \tan(x+\Delta x)\tan x}{\cos \Delta x}$$

which, using (1.2.2), shows that

$$\lim_{\Delta x \to 0} \frac{\tan(x+\Delta x) - \tan x}{\Delta x} = \Big(\lim_{\Delta x \to 0} \frac{\sin \Delta x}{\Delta x}\Big)\Big(\lim_{\Delta x \to 0} \frac{1 + \tan(x+\Delta x)\tan x}{\cos \Delta x}\Big) = \sec^2 x\,.$$

• **Higher-order derivatives**:

Let f be defined on an open interval $I = (a, b)$. If f' exists on I and possesses derivatives at every point in I, by definition we use f'' to denote the derivative of f'.In other words,

$$f''(x) = \frac{d}{dx}f'(x) = \frac{d}{dx}\frac{d}{dx}f(x) \equiv \frac{d^2}{dx^2}f(x) = \frac{d^2 f(x)}{dx^2}\Big(= \frac{d^2 y}{dx^2} \text{ if } y = f(x)\Big)\,.$$

The function f'' is called the second derivative of f. Similar as the "first" derivative case, $f''(c) = \frac{d^2}{dx^2}\Big|_{x=c} f(x)$.

The third derivatives and even higher-order derivatives are denoted by the following: if $y = f(x)$,

$$\begin{array}{rccc} \text{Third derivative: } y''' & f'''(x) & \frac{d^3}{dx^3}f(x) & \frac{d^3 f(x)}{dx^3} \\ \text{Fourth derivative: } y^{(4)} & f^{(4)}(x) & \frac{d^4}{dx^4}f(x) & \frac{d^4 f(x)}{dx^4} \\ & \vdots & & \\ \text{n-th derivative: } y^{(n)} & f^{(n)}(x) & \frac{d^n}{dx^n}f(x) & \frac{d^n f(x)}{dx^n}\,. \end{array}$$

2.3 The Chain Rule

The chain rule is used to study the derivative of composite functions.

Theorem 2.20: Chain Rule - 連鎖律

Let I, J be open intervals, $f : J \to \mathbb{R}$, $g : I \to \mathbb{R}$ be real-valued functions, and the range of g is contained in J. If g is differentiable at $c \in I$ and f is differentiable at $g(c)$, then $f \circ g$ is differentiable at c and

$$\frac{d}{dx}\Big|_{x=c}(f \circ g)(x) = f'(g(c))g'(c)\,.$$

Proof. To simplify the notation, we set $d = g(c)$.

Let $\varepsilon > 0$ be given. Since f is differentiable at d and g is differentiable at c, there exist $\delta_1, \delta_2 > 0$ such that

$$\Big|\frac{f(d+k) - f(d)}{k} - f'(d)\Big| < \frac{\varepsilon}{2(1 + |g'(c)|)} \quad \text{whenever} \quad 0 < |k| < \delta_1\,,$$
$$\Big|\frac{g(c+h) - g(c)}{h} - g'(c)\Big| < \min\Big\{1, \frac{\varepsilon}{2(1 + |f'(d)|)}\Big\} \quad \text{whenever} \quad 0 < |h| < \delta_2\,.$$

Therefore,

$$\big|f(d+k) - f(d) - f'(d)k\big| \leqslant \frac{\varepsilon}{2(1 + |g'(c)|)}|k| \quad \text{whenever} \quad |k| < \delta_1\,,$$
$$\big|g(c+h) - g(c) - g'(c)h\big| \leqslant \min\Big\{1, \frac{\varepsilon}{2(1 + |f'(d)|)}\Big\}|h| \quad \text{whenever} \quad |h| < \delta_2\,.$$

By Theorem 2.10, g is continuous at c; thus $\lim\limits_{h\to 0} g(c+h) = g(c)$. This fact provides $\delta_3 > 0$ such that

$$\big|g(c+h) - g(c)\big| < \delta_1 \quad \text{whenever} \quad |h| < \delta_3\,.$$

Define $\delta = \min\{\delta_2, \delta_3\}$. Then $\delta > 0$. Moreover, if $|h| < \delta$, the number $k \equiv g(c+h) - g(c)$ satisfies $|k| < \delta_1$. As a consequence, if $|h| < \delta$,

$$\begin{aligned}
&\big|(f\circ g)(c+h) - (f\circ g)(c) - f'(d)g'(c)h\big| = \big|f(g(c+h)) - f(d) - f'(d)g'(c)h\big| \\
&\qquad = \big|f(d+k) - f(d) - f'(d)g'(c)h\big| \\
&\qquad = \big|f(d+k) - f(d) - f'(d)k + f'(d)k - f'(d)g'(c)h\big| \\
&\qquad \leqslant \big|f(d+k) - f(d) - f'(d)k\big| + \big|f'(d)\big|\big|k - g'(c)h\big| \\
&\qquad \leqslant \frac{\varepsilon}{2(1+|g'(c)|)}|k| + \big|f'(d)\big|\big|g(c+h) - g(c) - g'(c)h\big| \\
&\qquad \leqslant \frac{\varepsilon}{2(1+|g'(c)|)}\big(|k - g'(c)h| + |g'(c)||h|\big) + \big|f'(d)\big|\frac{\varepsilon}{2(1+|f'(d)|)} \\
&\qquad \leqslant \frac{\varepsilon}{2(1+|g'(c)|)}\big(|h| + |g'(c)||h|\big) + \big|f'(d)\big|\frac{\varepsilon|h|}{2(1+|f'(d)|)} \\
&\qquad = \frac{\varepsilon}{2}|h| + \frac{\big|f'(d)\big|}{2(1+|f'(d)|)}\varepsilon|h|\,.
\end{aligned}$$

The inequality above implies that if $0 < |h| < \delta$,

$$\Big|\frac{(f\circ g)(c+h) - (f\circ g)(c)}{h} - f'(d)g'(c)\Big| \leqslant \frac{\varepsilon}{2} + \frac{\big|f'(d)\big|}{2(1+|f'(d)|)}\varepsilon < \varepsilon$$

which concludes the chain rule. $\square$

How to memorize the chain rule? Let $y = g(x)$ and $u = f(y)$. Then the derivative $u = (f\circ g)(x)$ is $\dfrac{du}{dx} = \dfrac{du}{dy}\dfrac{dy}{dx}$.

Example 2.21. Let $f(x) = (3x - 2x^2)^3$. Then $f'(x) = 3(3x - 2x^2)^2(3 - 4x)$.

Example 2.22. Let $f(x) = \Big(\dfrac{3x-1}{x^2+3}\Big)^2$. Then

$$\begin{aligned}
f'(x) &= 2\Big(\frac{3x-1}{x^2+3}\Big)^{2-1}\frac{d}{dx}\frac{3x-1}{x^2+3} = \frac{2(3x-1)}{x^2+3}\cdot\frac{3(x^2+3) - 2x(3x-1)}{(x^2+3)^2} \\
&= \frac{2(3x-1)(-3x^2+2x+9)}{(x^2+3)^3}\,.
\end{aligned}$$

Example 2.23. Let $f(x) = \tan^3\left[(x^2-1)^2\right]$. Then

$$\begin{aligned} f'(x) &= \left\{3\tan^2\left[(x^2-1)^2\right]\sec^2\left[(x^2-1)^2\right]\right\} \times \left[2(x^2-1)\cdot(2x)\right] \\ &= 12x(x^2-1)\tan^2\left[(x^2-1)^2\right]\sec^2\left[(x^2-1)^2\right]. \end{aligned}$$

Example 2.24. Let $f:\mathbb{R}\to\mathbb{R}$ be defined by

$$f(x) = \begin{cases} x^2\sin\dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0\,. \end{cases}$$

Then if $x \neq 0$, by the chain rule we have

$$\begin{aligned} f'(x) &= \Big(\frac{d}{dx}x^2\Big)\sin\frac{1}{x} + x^2\Big(\frac{d}{dx}\sin\frac{1}{x}\Big) = 2x\sin\frac{1}{x} + x^2\cos\frac{1}{x}\Big(\frac{d}{dx}\frac{1}{x}\Big) \\ &= 2x\sin\frac{1}{x} + x^2\cos\frac{1}{x}\Big(-\frac{1}{x^2}\Big) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}\,. \end{aligned}$$

Next we compute $f'(0)$. If $\Delta x \neq 0$, we have

$$\Big|\frac{f(\Delta x)-f(0)}{\Delta x}\Big| = \Big|\Delta x\sin\frac{1}{\Delta x}\Big| \leqslant |\Delta x|\,;$$

thus $-|\Delta x| \leqslant \dfrac{f(\Delta x)-f(0)}{\Delta x} \leqslant |\Delta x|$ for all $\Delta x \neq 0$ and the Squeeze Theorem implies that

$$f'(0) = \lim_{\Delta x\to 0}\frac{f(\Delta x)-f(0)}{\Delta x} = 0\,.$$

Therefore, we conclude that

$$f'(x) = \begin{cases} 2x\sin\dfrac{1}{x} - \cos\dfrac{1}{x} & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$

Definition 2.25

Let f be a function defined on an open interval I. f is said to be continuously differentiable on I if f is differentiable on I and f' is continuous on I.

The function f given in Example 2.24 is differentiable on $\mathbb{R}$ but not continuously differentiable since $\lim\limits_{x\to 0} f'(x)$ D.N.E.

2.4 Implicit Differentiation

An implicit function is a function that is defined implicitly by an equation that x and y satisfy, by associating one of the variables (the value y) with the others (the arguments x). For example, $x^2 + y^2 = 1$ and $x = \cos y$ are implicit functions. Sometimes we know how to express y in terms of x from the equation (such as the first case above $y = \sqrt{1-x^2}$ or $y = -\sqrt{1-x^2}$), while in most cases there is no way to know what the function y of x exactly is.

Given an implicit function (without solving for y in terms of x from the equation), can we find the derivative of y? This is the main topic of this section. We first focus on implicit functions of the form $f(x) = g(y)$. If $f(a) = g(b)$, we are interested in how the set $\big\{(x,y) \,\big|\, f(x) = g(y)\big\}$ looks like "mathematically" near (a,b).

Theorem 2.26: Implicit Function Theorem - 隱函數定理簡單版

Let f, g be continuously differentiable functions defined on some open intervals, and $f(a) = g(b)$. If $g'(b) \neq 0$, then there exists a unique continuously differentiable function $y = h(x)$, defined in an open interval containing a, satisfying that $b = h(a)$ and $f(x) = g(h(x))$.

Example 2.27. Let us compute the derivative of $h(x) = x^r$, where $r = \dfrac{p}{q}$ for some $p, q \in \mathbb{N}$ and $(p,q) = 1$. Write $y = h(x)$. Then $y^q = x^p$. Since $\dfrac{d}{dy} y^q = qy^{q-1} \neq 0$ if $y \neq 0$, by the Implicit Function Theorem we find that h is differentiable at every x satisfying $x \neq 0$. Since $h(x)^q = x^p$, by the chain rule we find that

$$qh(x)^{q-1}h'(x) = px^{p-1} \qquad \forall\, x \neq 0\,;$$

thus

$$h'(x) = \frac{p}{q} h(x)^{1-q} x^{p-1} = \frac{p}{q} x^{\frac{p}{q}(1-q)+p-1} = rx^{r-1} \qquad \forall\, x \neq 0\,.$$

If r is a negative rational number, we can apply the quotient and find that

$$\frac{d}{dx} x^r = \frac{d}{dx} \frac{1}{x^{-r}} = \frac{rx^{-r-1}}{x^{-2r}} = rx^{r-1} \qquad \forall\, x \neq 0\,.$$

Therefore, we conclude that

$$\frac{d}{dx} x^r = rx^{r-1} \qquad \forall\, x \neq 0\,. \tag{2.4.1}$$

Remark 2.28. The derivative of x^r can also be computed by first finding the derivative of $x^{\frac{1}{p}}$ $\big($that is, find the limit $\lim\limits_{\Delta x\to 0}\dfrac{(x+\Delta x)^{\frac{1}{p}}-x^{\frac{1}{p}}}{\Delta x}\big)$ and then apply the chain rule.

Example 2.29. Suppose that y is an implicit function of x given that $y^3+y^2-5y-x^2=-4$.

1. Find $\dfrac{dy}{dx}$.
2. Find the tangent line passing through the point $(3,-1)$.

Let $f(x)=x^2-4$ and $g(y)=y^3+y^2-5y$. Then $g'(y)=3y^2+2y-5$; thus if $y\neq 1$ or $y\neq -\dfrac{5}{3}$ $\big($or equivalently, $x\neq \pm 1$ or $x\neq \pm\sqrt{\dfrac{283}{27}}\big)$,

$$\frac{dy}{dx}=\frac{2x}{3y^2+2y-5}\,.$$

Since $(1,-3)$ satisfies the relation $y^3+y^2-5y-x^2=-4$, the slope of the tangent line passing through $(3,-1)$ is $\dfrac{2\cdot 3}{3(-1)^2+2(-1)-5}=-\dfrac{3}{2}$; thus the desired tangent line is

$$y=-\frac{3}{2}(x-3)-1\,.$$

Example 2.30. Find $\dfrac{dy}{dx}$ implicitly for the equation $\sin y=x$.

Let $f(x)=x$ and $g(y)=\sin y$. Then $g'(y)=\cos y$; thus if $y\neq n\pi+\dfrac{\pi}{2}$ (or equivalently, $x\neq\pm 1$),

$$\frac{dy}{dx}=\frac{1}{\cos y}\,. \tag{2.4.2}$$

Similarly, for function y defined implicitly by $\cos y=x$, we find that if $y\neq n\pi$ (or equivalently, $x\neq\pm 1$),

$$\frac{dy}{dx}=-\frac{1}{\sin y}\,. \tag{2.4.3}$$

Remark 2.31. The curve consisting of points (x,y) satisfying the relation $\sin y=x$ cannot be the graph of a function since one x may corresponds to several y; however, the curve consisting of points (x,y) satisfying the relation $\sin y=x$ as well as $-\dfrac{\pi}{2}<y<\dfrac{\pi}{2}$ is the graph of a function called arcsin. In other words, for each $x\in(-1,1)$, there exists a unique $y\in\big(-\dfrac{\pi}{2},\dfrac{\pi}{2}\big)$ satisfying $\sin y=x$, and such y is denoted by $\arcsin x$. Since for $y\in\big(-\dfrac{\pi}{2},\dfrac{\pi}{2}\big)$ we must have $\cos y>0$, by the fact that $\sin^2 y+\cos^2 y=1$, using (2.4.2) we find that

$$\frac{d}{dx}\arcsin x=\frac{1}{\sqrt{1-x^2}}\qquad\forall\, x\in(-1,1)\,. \tag{2.4.4}$$

Similarly, the curve consisting of points (x, y) satisfying the relation $\cos y = x$ as well as $0 < y < \pi$ is the graph of a function called arccos, and (2.4.3) implies that

$$\frac{d}{dx} \arccos x = -\frac{1}{\sqrt{1-x^2}} \qquad \forall\, x \in (-1, 1)\,. \tag{2.4.5}$$

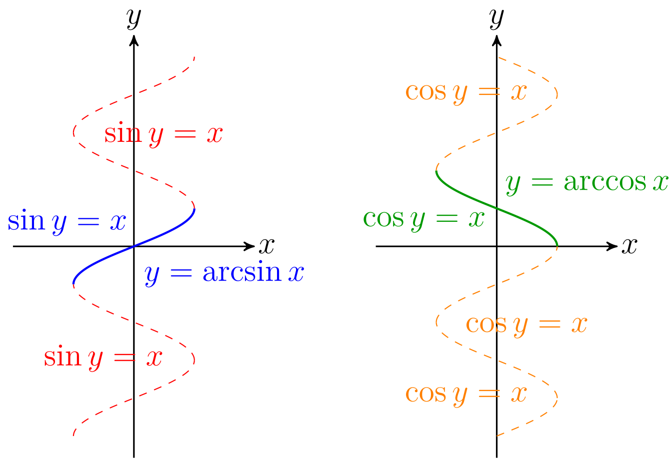

Figure 2.1: The graph of functions $y = \arcsin x$ and $y = \arccos x$

There are, unfortunately, many implicit functions that are not given by the equation of the form $f(x) = g(y)$. Nevertheless, there is a more powerful version of the Implicit Function Theorem that guarantees the continuous differentiability of the implicit functions defined through complicated relations between x and y (written in the form $f(x, y) = 0$). In the following, we always assume that the implicit function given by the equation that x and y satisfy is differentiable.

Example 2.32. Find the second derivative of the implicit function given by the equation $y = \cos(5x - 3y)$.

Differentiate in x once, we find that $\dfrac{dy}{dx} = -\sin(5x-3y) \cdot \big(5 - 3\dfrac{dy}{dx}\big)$; thus

$$\frac{dy}{dx} = \frac{-5\sin(5x-3y)}{1-3\sin(5x-3y)} = \frac{5}{3}\Big[1 - \frac{1}{1-3\sin(5x-3y)}\Big]\,. \tag{2.4.6}$$

Differentiate the equation above in x, we obtain that

$$\frac{d^2y}{dx^2} = -\frac{5}{3} \cdot \frac{3\cos(5x-3y)(5-3y')}{\big[1-3\sin(5x-3y)\big]^2} = -\frac{5\cos(5x-3y)(5-3y')}{\big[1-3\sin(5x-3y)\big]^2}$$

and (2.4.6) further implies that $\dfrac{d^2y}{dx^2} = -\dfrac{25\cos(5x-3y)}{\big[1-3\sin(5x-3y)\big]^3}$.

Example 2.33. Show that if it is possible to draw three normals from the point $(a, 0)$ to the parabola $x = y^2$, then $a > \frac{1}{2}$.

Suppose that the line L connecting $(a, 0)$ and (b^2, b), where $b \neq 0$, is normal to the parabola $x = y^2$. The derivative of the function defined implicitly by $x = y^2$ satisfies that

$$1 = 2y\frac{dy}{dx};$$

thus the slope of the tangent line passing through (b^2, b) is $\frac{1}{2b}$. Since line L is perpendicular to the tangent line passing through (b^2, b), we must have

$$\frac{1}{2b} \cdot \frac{b - 0}{b^2 - a} = -1.$$

Therefore, $a = \frac{1}{2} + b^2$. Since $b \neq 0$, $a > \frac{1}{2}$.

Chapter 3

Applications of Differentiation

3.1 Extrema on an Interval

Definition 3.1

Let f be defined on an interval I containing c.

1. $f(c)$ is the minimum of f on I when $f(c) \leqslant f(x)$ for all x in I.
2. $f(c)$ is the maximum of f on I when $f(c) \geqslant f(x)$ for all x in I.

The minimum and maximum of a function on an interval are the extreme values, or extrema (the singular form of extrema is extremum), of the function on the interval. The minimum and maximum of a function on an interval are also called the absolute minimum and absolute maximum, or the global minimum and global maximum, on the interval. Extrema can occur at interior points or end-points of an interval. Extrema that occur at the end-points are called end-point extrema.

Theorem 3.2: Extreme Value Theorem - 極值定理

If f is continuous on a closed interval $[a, b]$, then f has both a minimum and a maximum on the interval.（連續函數在閉區間上必有最大最小值）

When f is continuous on an open interval (a, b) (or a half-open half-closed interval), it is still possibly that f attains its maximum or minimum but there is no guarantee. Moreover, it is also possible that f does not attain its extrema when f is continuous on an interval which is not closed.

Definition 3.3

Let f be defined on an interval I containing c.

1. If there is an open interval containing c on which $f(c)$ is a maximum, then $f(c)$ is called a relative maximum of f, or you can say that f has a relative maximum at $\big(c, f(c)\big)$.

2. If there is an open interval containing c on which $f(c)$ is a minimum, then $f(c)$ is called a relative minimum of f, or you can say that f has a relative minimum at $\big(c, f(c)\big)$.

The plural of relative maximum is relative maxima, and the plural of relative minimum is relative minima. Relative maximum and relative minimum are sometimes called local maximum and local minimum, respectively.

Definition 3.4

Let f be defined on an open interval containing c. The number/point c is called a critical number or critical point of f if $f'(c) = 0$ or if f is not differentiable at c.

Theorem 3.5

If f has a relative minimum or relative maximum at $x = c$, then c is a critical point of f.

Proof. W.L.O.G., we assume that f is differentiable at c. If $f'(c) > 0$, then there exists $\delta_1 > 0$ such that

$$\Big|\frac{f(x) - f(c)}{x - c} - f'(c)\Big| < \frac{f'(c)}{2} \qquad \text{if} \quad 0 < |x - c| < \delta_1\,;$$

thus

$$\frac{f'(c)}{2} < \frac{f(x) - f(c)}{x - c} < \frac{3f'(c)}{2} \qquad \text{if} \quad 0 < |x - c| < \delta_1\,.$$

1. If $0 < x - c < \delta_1$,

$$f(c) + \frac{f'(c)}{2}(x - c) < f(x) < f(c) + \frac{3f'(c)}{2}(x - c)$$

which implies that f cannot attain a relative maximum at $x = c$ since $f(x) > f(c)$ on the right-hand side of c.

2. if $-\delta < x - c < 0$,

$$f(c) + \frac{f'(c)}{2}(x-c) > f(x) > f(c) + \frac{3f'(c)}{2}(x-c)$$

which implies that f cannot attain a relative minimum at $x = c$ since $f(c) > f(x)$ on the left-hand side of c.

Therefore, we conclude that if $f'(c) > 0$, then f cannot attain either a relative maximum or minimum at $x = c$. Similar conclusion can be drawn for the case $f'(c) < 0$; thus if f attains a relative extremum at $x = c$, then $f'(c) = 0$. $\square$

Remark 3.6. A more strict version of Theorem 3.5 is called ***Fermat's Theorem*** which is stated as follows:

> If f has a local maximum or minimum at c, and if $f'(c)$ exists, then $f'(c) = 0$.

The way to find extrema of a continuous function f on a closed interval $[a, b]$:

1. Find the critical points of f in (a, b).
2. Evaluate f at each critical points in (a, b).
3. Evaluate f at the end-points of $[a, b]$.
4. The least of these values is the minimum, and the greatest is the maximum.

Example 3.7. Find the extrema of $f(x) = 2\sin x - \cos 2x$ on the interval $[0, 2\pi]$.

Since f is differentiable on $(0, 2\pi)$, a critical point c satisfies

$$0 = f'(c) = 2\cos c + 2\sin 2c = 2\cos c(1 + 2\sin c)\,.$$

Therefore, $c = \frac{\pi}{2}$, $c = \frac{3\pi}{2}$, $c = \frac{7\pi}{6}$ or $c = \frac{11\pi}{6}$, and the values of f at these critical points are

$$\begin{aligned} f\big(\tfrac{\pi}{2}\big) &= 2\cdot 1 - (-1) = 3\,, & f\big(\tfrac{3\pi}{2}\big) &= 2\cdot(-1) - (-1) = -1\,, \\ f\big(\tfrac{7\pi}{6}\big) &= 2\cdot\big(-\tfrac{1}{2}\big) - \tfrac{1}{2} = -\tfrac{3}{2}\,, & f\big(\tfrac{11\pi}{6}\big) &= 2\cdot\big(-\tfrac{1}{2}\big) - \tfrac{1}{2} = -\tfrac{3}{2}\,. \end{aligned}$$

On the other hand, the values of f at the end-points are

$$f(0) = 2\cdot 0 - 1 = -1 \quad \text{and} \quad f(2\pi) = 2\cdot 0 - 1 = -1\,.$$

Therefore, $f\big(\frac{\pi}{2}\big) = 3$ is the maximum of f on $[0, 2\pi]$, while the minimum of f on $[0, 2\pi]$ occurs at $c = \frac{7\pi}{6}$ and $c = \frac{11\pi}{6}$ and the minimum is $-\frac{3}{2}$.

3.2 Rolle's Theorem and the Mean Value Theorem

Theorem 3.8: Rolle's Theorem

Let $f : [a, b] \to \mathbb{R}$ be a continuous function and f is differentiable on (a, b). If $f(a) = f(b)$, then there is at least one point $c \in (a, b)$ such that $f'(c) = 0$.

Proof. If f is a constant function, then $f'(x) = 0$ for all $x \in (a, b)$. Now suppose that f is not a constant function on $[a, b]$, by the Extreme Value Theorem implies that f has a maximum and a minimum on $[a, b]$, and the maximum and the minimum of f on $[a, b]$ are different. Therefore, there must be a point $c \in (a, b)$ at which f attains its extreme value. By Theorem 3.5, $f'(c) = 0$. $\square$

Theorem 3.9: Mean Value Theorem

If $f : [a, b] \to \mathbb{R}$ is continuous and f is differentiable on (a, b), then there exists a point $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Proof. Define $g : [a, b] \to \mathbb{R}$ by $g(x) = \big[f(x) - f(a)\big](b - a) - \big[f(b) - f(a)\big](x - a)$. Then $g : [a, b] \to \mathbb{R}$ is continuous and g is differentiable on (a, b). Moreover, $g(a) = g(b) = 0$; thus the Rolle Theorem implies that there exists $c \in (a, b)$ such that $g'(c) = 0$. On the other hand,

$$0 = g'(c) = (b - a)f'(c) - \big[f(b) - f(a)\big];$$

thus there exists $c \in (a, b)$ satisfying $f'(c) = \dfrac{f(b) - f(a)}{b - a}$. $\square$

Remark 3.10. In fact, by modifying the proof of the mean value theorem a little bit, we can show the following: Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $g'(x) \neq 0$ for all $x \in (a, b)$, then there exists $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}.$$

The statement above is a generalization of the mean value theorem and is called the Cauchy mean value theorem (see Theorem **??**).

Example 3.11. Note that the sine function is continuous on any closed interval $[a, b]$ and is differentiable on (a, b). Therefore, the mean value theorem implies that there exists $c \in (a, b)$ such that

$$\cos c = \frac{d}{dx}\Big|_{x=c} \sin x = \frac{\sin b - \sin a}{b - a}$$

which implies that $|\sin a - \sin b| = |\cos c||a - b| \leqslant |a - b|$. Therefore,

$$|\sin x - \sin y| \leqslant |x - y| \qquad \forall\, x, y \in \mathbb{R}\,.$$

Similarly,

$$|\cos x - \cos y| \leqslant |x - y| \qquad \forall\, x, y \in \mathbb{R}\,.$$

3.3 Monotone Functions and the First Derivative Test

Definition 3.12

Let f be defined on an interval I.

1. f is said to be increasing on I if

$$f(x_1) \leqslant f(x_2) \qquad \forall\, x_1, x_2 \in I \text{ and } x_1 < x_2\,.$$

2. f is said to be decreasing on I if

$$f(x_1) \geqslant f(x_2) \qquad \forall\, x_1, x_2 \in I \text{ and } x_1 < x_2\,.$$

3. f is said to be strictly increasing on I if

$$f(x_1) < f(x_2) \qquad \forall\, x_1, x_2 \in I \text{ and } x_1 < x_2\,.$$

4. f is said to be strictly decreasing on I if

$$f(x_1) > f(x_2) \qquad \forall\, x_1, x_2 \in I \text{ and } x_1 < x_2\,.$$

When f is either increasing on I or decreasing on I, then f is said to be monotone. When f is either strictly increasing on I or strictly decreasing on I, then f is said to be strictly monotone on I.

Remark 3.13. Note that f is increasing on I if

$$\frac{f(x_1) - f(x_2)}{x_1 - x_2} \geqslant 0 \qquad \forall\, x_1, x_2 \in I \text{ and } x_1 \neq x_2\,.$$

Therefore, f is increasing on I if the slope of each secant line of the graph of f is non-negative. Similar conclusions hold for the other cases.

Example 3.14. The function $f(x) = x^3$ is strictly increasing on $\mathbb{R}$, and $f(x) = -x^3$ is strictly decreasing on $\mathbb{R}$.

Example 3.15. The sine function is strictly increasing on $\left[2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right]$ for all $n \in \mathbb{Z}$, but decreasing on $\left[2n\pi - \frac{\pi}{2}, 2n\pi + \frac{3\pi}{2}\right]$ for all $n \in \mathbb{Z}$. However, the sine function is **not** strictly increasing on $\bigcup_{n=-\infty}^{\infty} \left[2n\pi - \frac{\pi}{2}, 2n\pi + \frac{\pi}{2}\right]$ and is **not** strictly decreasing on $\bigcup_{n=-\infty}^{\infty} \left[2n\pi - \frac{\pi}{2}, 2n\pi + \frac{3\pi}{2}\right]$.

Theorem 3.16

Let $f : [a, b] \to \mathbb{R}$ be continuous and f is differentiable on (a, b).

1. If $f'(x) \geqslant 0$ for all $x \in (a, b)$, then f is increasing on $[a, b]$.
2. If $f'(x) \leqslant 0$ for all $x \in (a, b)$, then f is decreasing on $[a, b]$.
3. If $f'(x) > 0$ for all $x \in (a, b)$, then f is strictly increasing on $[a, b]$.
4. If $f'(x) < 0$ for all $x \in (a, b)$, then f is strictly decreasing on $[a, b]$.

Proof. We only prove 1 since all the other conclusion can be proved in a similar fashion.

Suppose that $f'(x) \geqslant 0$, and $x_1 < x_2$. By the Mean Value Theorem, there exists $c \in (x_1, x_2)$ such that
$$\frac{f(x_1) - f(x_2)}{x_1 - x_2} = f'(c) \geqslant 0\,;$$
thus $f(x_1) \leqslant f(x_2)$ if $x_1 < x_2$. □

Remark 3.17. The condition $f'(x) > 0$ is just a sufficient condition for that f is strictly increasing, but not a necessary condition. For example, $f(x) = x^3$ is strictly increasing on $\mathbb{R}$, but $f'(0) = 0$.

Example 3.18. Show that
$$\cos x \geqslant 1 - \frac{x^2}{2} \qquad \forall\, x \geqslant 0\,. \tag{3.3.1}$$

Let $f(x) = \cos x - 1 + \frac{x^2}{2}$. In order to show (3.3.1), we need to show that $f(x) \geqslant 0$ for all $x \geqslant 0$. Since $f'(x) = -\sin x + x$, by Theorem 0.13 we find that f' is non-negative on $[0, \infty)$. Therefore, Theorem 3.16 implies that f is increasing on $[0, \infty)$ which further shows that $f(x) \geqslant f(0) = 0$ for all $x \geqslant 0$.

Example 3.19. Using (3.3.1), we can show that

$$\sin x \geqslant x - \frac{x^3}{6} \qquad \forall\, x \geqslant 0\,.$$

In fact, by defining $g(x) = \sin x - x + \dfrac{x^3}{6}$, using (3.3.1) we find that

$$g'(x) = \cos x - 1 + \frac{x^2}{2} \geqslant 0 \qquad \forall\, x \geqslant 0\,;$$

thus g is increasing on $[0, \infty)$ which shows that $g(x) \geqslant g(0) = 0$ for all $x \geqslant 0$. Similar argument then shows that

$$\cos x \leqslant 1 - \frac{x^2}{2} + \frac{x^4}{24} \qquad \forall\, x \geqslant 0$$

and the inequality above in turn implies that

$$\sin x \leqslant x - \frac{x^3}{6} + \frac{x^5}{120} \qquad \forall\, x \geqslant 0\,.$$

By induction, we can show that for all $k \in \mathbb{N} \cup \{0\}$,

$$x - \frac{x^3}{3!} + \cdots + \frac{x^{4k+1}}{(4k+1)!} - \frac{x^{4k+3}}{(4k+3)!} \leqslant \sin x \leqslant x - \frac{x^3}{3!} + \cdots + \frac{x^{4k+1}}{(4k+1)!} \qquad \forall\, x \geqslant 0\,,$$

$$1 - \frac{x^2}{2!} + \cdots + \frac{x^{4k}}{(4k)!} - \frac{x^{4k+2}}{(4k+2)!} \leqslant \cos x \leqslant 1 - \frac{x^2}{2} + \cdots + \frac{x^{4k}}{(4k)!} \qquad \forall\, x \geqslant 0\,.$$

Theorem 3.20: The First Derivative Test

Let f be a continuous function defined on an open interval I containing c. If f is differentiable on I, except possibly at c, then

1. If f' changes from negative to positive at c, then $f(c)$ is a local minimum of f.
2. If f' changes from positive to negative at c, then $f(c)$ is a local maximum of f.
3. If f' is sign definite on $I\backslash\{c\}$, then $f(c)$ is neither a relative minimum or relative maximum of f.

Proof. We only prove 1. Assume that f' changes from negative to positive at c. Then there exists a and b in I such that

$$f'(x) < 0 \text{ for all } x \in (a, c) \quad \text{and} \quad f'(x) > 0 \text{ for all } x \in (c, b)\,.$$

By Theorem 3.16, f is decreasing on (a, c) and is increasing on (c, b). Therefore, $f(c)$ is a minimum on an open interval (a, b); thus is a relative minimum on I. $\square$

Example 3.21. Find the relative extrema of $f(x) = \frac{1}{2}x - \sin x$ in the interval $(0, 2\pi)$.

By Theorem 3.5 the relative extrema occurs at critical points. Since f is differentiable on $(0, 2\pi)$, a critical point x satisfies

$$0 = f'(x) = \frac{1}{2} - \cos x$$

which implies that $c = \frac{\pi}{3}$ and $c = \frac{5\pi}{3}$ are the only critical points. To determine if $f\left(\frac{\pi}{3}\right)$ or $f\left(\frac{5\pi}{3}\right)$ is a relative minimum, we apply Theorem 3.20 and found that, since f' changes from negative to positive at $\frac{\pi}{3}$ and changes from positive to negative at $\frac{5\pi}{3}$, $f\left(\frac{\pi}{3}\right)$ is a relative minimum of f on $(0, 2\pi)$.

Remark 3.22. When a differentiable function f attains a local minimum at an interior point c, it is not necessary that f' changes from positive to negative. For example, consider the function $f : \mathbb{R} \to \mathbb{R}$ defined by

$$f(x) = \begin{cases} x^2\left(1 + \sin\frac{1}{x}\right) & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$

Then

$$f'(x) = \begin{cases} 2x\left(1 + \sin\frac{1}{x}\right) - \cos\frac{1}{x} & \text{if } x \neq 0\,, \\ 0 & \text{if } x = 0\,. \end{cases}$$

Therefore,

1. 0 is a critical point of f.
2. f attains a (global) minimum at 0 since obviously $f(x) \geqslant 0 = f(0)$ for all $x \in \mathbb{R}$.
3. It is impossible to determine if f' changes "from negative to positive" or "from positive to negative" at 0.

3.4 Concavity（凹性）and the Second Derivative Test

Definition 3.23

Let f be differentiable on an open interval I. The graph of f is concave upward（凹向上）on I if f' is strictly increasing on the interval and concave downward（凹向下）on I if f' is strictly decreasing on the interval.

Remark 3.24. It does not really matter if f' has to be strictly monotone, instead of just monotone, in order to define the concavity of the graph of f. Here we define the concavity by the strict monotonicity.

- Graphical interpretation of concavity: Let f be differentiable on an open interval I.

  1. If the graph of f is concave upward on I, then the graph of f lies above all of its tangent lines on I.
  2. If the graph of f is concave downward on I, then the graph of f lies below all of its tangent lines on I.

The following theorem is a direct consequence of Theorem 3.16.

Theorem 3.25: Test for Concavity

Let f be a twice differentiable function on an open interval I.

1. If $f''(x) > 0$ for all x in I, then the graph of f is concave upward on I.
2. If $f''(x) < 0$ for all x in I, then the graph of f is concave downward on I.

Example 3.26. Determine the open intervals on which the graph of $f(x) = \dfrac{6}{x^2+3}$ is concave upward or concave downward.

First we compute the second derivative of f:

$$f'(x) = \frac{-12x}{(x^2+3)^2} \Rightarrow f''(x) = -12\frac{(x^2+3)^2 - 2(x^2+3)(2x)x}{(x^2+3)^4} = \frac{36(x^2-1)}{(x^2+3)^3}.$$

Therefore, the graph of f is concave upward if $x > 1$ and is concave downward if $x < 1$.

Definition 3.27: Point of inflection（反曲點）

Let f be a differentiable function on an open interval containing c. The point $\big(c, f(c)\big)$ is called a point of inflection (or simply an inflection point) of the graph of f if the concavity of f changes from upward to downward or downward to upward at this point.

Example 3.28. Recall Example 3.26 $\Big(f(x) = \dfrac{6}{x^2+3}$ with $f''(x) = \dfrac{36(x^2-1)}{(x^2+3)^3}\Big)$. Since f'' changes sign at $x = \pm 1$, $\Big(\pm 1, \dfrac{3}{2}\Big)$ are both points of inflection of the graph of f.

Theorem 3.29

Let f be a differentiable function on an open interval containing c. If $\big(c, f(c)\big)$ is a point of inflection of the graph of f, then either $f''(c) = 0$ or $f''(c)$ does not exist.

Remark 3.30. A point $\big(c, f(c)\big)$ may not be an inflection point of the graph of f even if $f''(c) = 0$. For example, the point $(0, 0)$ is not an inflection point of $f(x) = x^4$ since $f''(x) > 0$ for all $x \neq 0$ which implies that the concavity of f does not change at $c = 0$.

Example 3.31. Determine the points of inflection and discuss the concavity of the graph of $f(x) = x^4 - 4x^3$. Note that the zero of f'' is $x = 0$ or $x = 2$ (since $f''(x) = 12x^2 - 24x$). Since $f''(x) > 0$ if $x < 0$ or $x > 2$, and $f''(x) > 0$ if $0 < x < 2$, we find that $(0, 0)$ and $(2, -16)$ are points of inflection of the graph of f.

Theorem 3.32

Let f be a twice differentiable function on an open interval I containing c, and c is a critical point of f.

1. If $f''(c) > 0$, then $f(c)$ is a relative minimum of f on I.
2. If $f''(c) < 0$, then $f(c)$ is a relative maximum of f on I.

Remark 3.33. If $f''(c) = 0$ for some critical point c of f, then f may have a relative maximum, a relative minimum, or neither at c. In such cases, you should use the First Derivative Test.

Proof of Theorem 3.32. Since $f''(c) > 0$, there exist $\delta > 0$ such that

$$\Big|\frac{f'(x) - f'(c)}{x - c} - f''(c)\Big| < \frac{f''(c)}{2} \quad \text{if } 0 < |x - c| < \delta.$$

Since c is a critical point of f, $f'(c) = 0$; thus the inequality above implies that

$$\frac{f''(c)}{2} < \frac{f'(x)}{x - c} < \frac{3f''(c)}{2} \quad \text{if } 0 < |x - c| < \delta.$$

In particular,

$$\begin{aligned} 0 < \frac{f'(c)}{2}(x - c) < f'(x) \qquad & \text{if } 0 < x - c < \delta\,, \\ f'(x) < \frac{f'(c)}{2}(x - c) < 0 \qquad & \text{if } -\delta < x - c < 0\,. \end{aligned}$$

Therefore, f' changes from negative to positive at c; thus $f(c)$ is a relative minimum of f on I. $\square$

Example 3.34. Recall Example 3.21 $\big(f(x) = \frac{1}{2}x - \sin x\big)$. We have established that $f\big(\frac{\pi}{3}\big)$ is a relative minimum of f on $(0, 2\pi)$ using the First Derivative Test. Note that $f''(x) = \sin x$; thus $f''\big(\frac{\pi}{3}\big) = \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2} > 0$. Therefore, without using the First Derivative Test, we can still conclude that $f\big(\frac{\pi}{3}\big)$ is a relative minimum of f on $(0, 2\pi)$ by the second derivative test.

Example 3.35. Show that for all $1 < p, q < \infty$ satisfying $\frac{1}{p} + \frac{1}{q} = 1$, we have

$$ab \leqslant \frac{a^p}{p} + \frac{b^q}{q} \qquad \forall\, a, b > 0\,. \tag{3.4.1}$$

The inequality above is called Young's inequality. We remark that if $\frac{1}{p} + \frac{1}{q} = 1$, then $q = \frac{p}{p-1}$.

For the moment we only show (3.4.1) for the case that $p, q \in \mathbb{Q}$ (because we have not talked about what it means by the p-th power if p is irrational). To show (3.4.1), we prove that for each given $b > 0$, the function $f : (0, \infty) \to \mathbb{R}$

$$f(x) \equiv \frac{x^p}{p} - bx + \frac{b^q}{q}$$

is non-negative. In other words, we have to show that the "minimum" of f is non-negative.

To find the minimum of f, we differentiate and find that $f'(x) = x^{p-1} - b$ which implies that $c = b^{\frac{1}{p-1}}$ is the only critical point. Since

$$f''(c) = (p-1)c^{p-2} = (p-1)b^{\frac{p-2}{p-1}} > 0\,,$$

the second derivative test implies that f attains a local minimum at c. Since there is no other critical points, f must attain its global minimum at c; thus

$$f(x) \geqslant f(c) \qquad \forall\, x \in (0, \infty)$$

and (3.4.1) is established since $f(c) = \dfrac{b^{\frac{p}{p-1}}}{p} - b^{1+\frac{1}{p-1}} + \dfrac{b^q}{q} = \dfrac{b^q}{p} - b^q + \dfrac{b^q}{q} = 0.$

Remark 3.36. Suppose that c is a critical point of a differentiable function f with $f''(c) = 0$. For f to attain a local extremum at c, $f'''(c)$ must be zero if the third derivative of f is continuous. If in addition $f^{(4)}$ is continuous, then

1. f attains a local maximum at c provided that $f^{(4)}(c) < 0$.

2. f attains a local minimum at c provided that $f^{(4)}(c) > 0$.

In general, if f is $2k$-times continuously differentiable (which means $f^{(2k)}$ exists everywhere and is continuous) and $f'(c) = f''(c) = \cdots = f^{(2k-1)}(c) = 0$, then

1. f attains a local maximum at c provided that $f^{(2k)}(c) < 0$.

2. f attains a local minimum at c provided that $f^{(2k)}(c) > 0$.

On the other hand, if f is $(2k+1)$-times continuously differentiable and $f'(c) = f''(c) = \cdots = f^{(2k)}(c) = 0$ but $f^{(2k+1)}(c) \neq 0$, then f cannot attain its local extremum at c.

3.5 A Summary of Curve Sketching

When sketching the graph of functions, you need to have the following on the plot.

1. x-intercepts and y-intercepts;

2. asymptotes;

3. absolution extrema and relative extrema;

4. points of inflection.

Example 3.37. Sketch the graph of the function $f(x) = \dfrac{3x-2}{\sqrt{2x^2+1}}$.

First, we note that the x-intercepts and y-intercepts are $\left(\frac{3}{2}, 0\right)$ and $(0, f(0)) = (0, -2)$. To determine the asymptotes, since $\sqrt{2x^2+1}$ are never zero, there is no vertical asymptote. As for the horizontal and slant asymptotes, by the fact that

$$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \frac{\frac{3x-2}{x}}{\frac{\sqrt{2x^2+1}}{x}} = \lim_{x\to\infty} \frac{3-\frac{2}{x}}{\sqrt{2+\frac{1}{x^2}}} = \lim_{y\to 0^+} \frac{3-2y}{\sqrt{2+y^2}} = \frac{3}{\sqrt{2}}$$

and

$$\lim_{x\to-\infty} f(x) = \lim_{x\to\infty} f(-x) = \lim_{x\to\infty} \frac{-3x-2}{\sqrt{2x^2+1}} = \lim_{x\to\infty} \frac{-3-\frac{2}{x}}{\sqrt{2+\frac{1}{x^2}}} = \lim_{y\to 0^+} \frac{3-2y}{-\sqrt{2+y^2}} = -\frac{3}{\sqrt{2}},$$

we find that there are two horizontal asymptotes $y = \pm\dfrac{3}{\sqrt{2}}$.

By the quotient rule,

$$\begin{aligned} f'(x) &= \frac{3\sqrt{2x^2+1} - (3x-2)\dfrac{d}{dx}(2x^2+1)^{\frac{1}{2}}}{2x^2+1} = \frac{3\sqrt{2x^2+1} - (3x-2)\dfrac{1}{2}(2x^2+1)^{-\frac{1}{2}} \cdot (4x)}{2x^2+1} \\ &= \frac{3(2x^2+1) - 2x(3x-2)}{(2x^2+1)^{\frac{3}{2}}} = \frac{4x+3}{(2x^2+1)^{\frac{3}{2}}} \end{aligned}$$

and

$$\begin{aligned} f''(x) &= \frac{4(2x^2+1)^{\frac{3}{2}} - (4x+3)\dfrac{3}{2}(2x^2+1)^{\frac{1}{2}} \cdot (4x)}{(2x^2+1)^3} = \frac{4(2x^2+1) - 6x(4x+3)}{(2x^2+1)^{\frac{5}{2}}} \\ &= \frac{-16x^2 - 18x + 4}{(2x^2+1)^{\frac{5}{2}}} = \frac{-2(8x^2+9x-2)}{(2x^2+1)^{\frac{5}{2}}}. \end{aligned}$$

Therefore, $x = -\dfrac{3}{4}$ is the only critical point and since f' changes from negative to positive at $-\dfrac{3}{4}$, $f\big(-\dfrac{3}{4}\big)$ is a relative minimum of f.

$f''(x) = 0$ occurs at $x_1 = \dfrac{-9-\sqrt{145}}{16}$ and $x_2 = \dfrac{-9+\sqrt{145}}{16}$. Since f'' changes sign at x_1 and x_2, $(x_1, f(x_1))$ and $(x_2, f(x_2))$ are inflection points of the graph of f.

3.6 Optimization Problems

Explanation of examples in Section 3.7 in the textbook:

1. 一製造商想設計一個底部為正方形、表面積 108 平方公分且上方有開口的箱子。要怎麼設計才能讓箱子容積最大？

2. Which points on the graph of $y = 4 - x^2$ are closest to the point $(0, 2)$? 拋物線 $y = 4 - x^2$ 上哪個點到 $(0, 2)$ 最近？

3. 試找出最小面積的方形頁面使之能上下留白三公分、左右留白兩公分且要包含 216 平方公分的長方形區域可用於印刷。

4. 兩根分別為 12 公尺及 28 公尺高的桿子相距 30 公尺。找出地面上一點使之到兩桿之頂端之距離和最小。

5. Four meters of wire is to be used to form a square and a circle. How much of the wire should be used for the square and how much should be used for the circle to enclose the maximum total area? 一總長 4 公尺的線要被分為兩段用來圍出一個正方形和一個圓形。要怎麼分段才能圍出最大的面積。

6. **Application in Physics**: Let v_1 be the velocity of light in air and v_2 the velocity of light in water. According to Fermat's Principle, a ray of light will travel from a point A in the air to a point B in the water by a path ACB that minimizes the time taken. Show that
$$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2},$$
where θ_1 (the angle of incidence) and θ_2 (the angle of refraction) are as shown. This equation is known as Snell's Law.

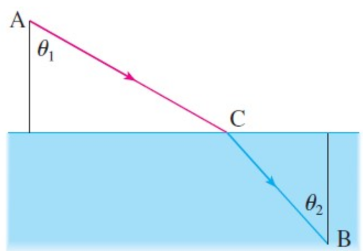

Figure 3.1: Snell's law

Proof. Assume that $A = (0, a)$ and $B = (b, -c)$. The goal is to find $C = (x, 0)$ so that
$$f(x) = \frac{\sqrt{x^2 + a^2}}{v_1} + \frac{\sqrt{(x-b)^2 + c^2}}{v_2}$$
is minimized. Differentiating f, we find that a critical point x of f satisfies
$$\frac{1}{v_1}\frac{x}{\sqrt{x^2 + a^2}} = \frac{1}{v_2}\frac{b-x}{\sqrt{(x-b)^2 + c^2}}.$$
Snell's law then is concluded from the fact that $\sin\theta_1 = \dfrac{x}{\sqrt{x^2 + a^2}}$ and $\sin\theta_2 = \dfrac{b-x}{\sqrt{(b-x)^2 + c^2}}$. □

7. **Application in Economics**: Suppose that

$$r(x) = \text{the revenue from selling } x \text{ items},$$
$$c(x) = \text{the cost of producing the } x \text{ items},$$
$$p(x) = r(x) - c(x) = \text{the profit from producing and selling } x \text{ items}.$$

Although x is usually an integer in many applications, we can learn about the behavior of these functions by defining them for all nonzero real numbers and by assuming they are differentiable functions. Economists use the terms *marginal revenue* (邊際收益), *marginal cost* (邊際成本), and *marginal profit* (邊際利潤) to name the derivatives $r'(x)$, $c'(x)$, and $p'(x)$ of the revenue, cost, and profit functions. Let us consider the relationship of the profit p to these derivatives. If $r(x)$ and $c(x)$ are differentiable for x in some interval of production possibilities, and if $p(x) = r(x) - c(x)$ has a maximum value there, it occurs at a critical point of $p(x)$ or at an end-point of the interval. If it occurs at a critical point, then $p'(x) = r'(x) - c'(x) = 0$ and we see that $r'(x) = c'(x)$. In economic terms, this last equation means that

> At a production level yielding maximum profit, marginal revenue equals marginal cost.

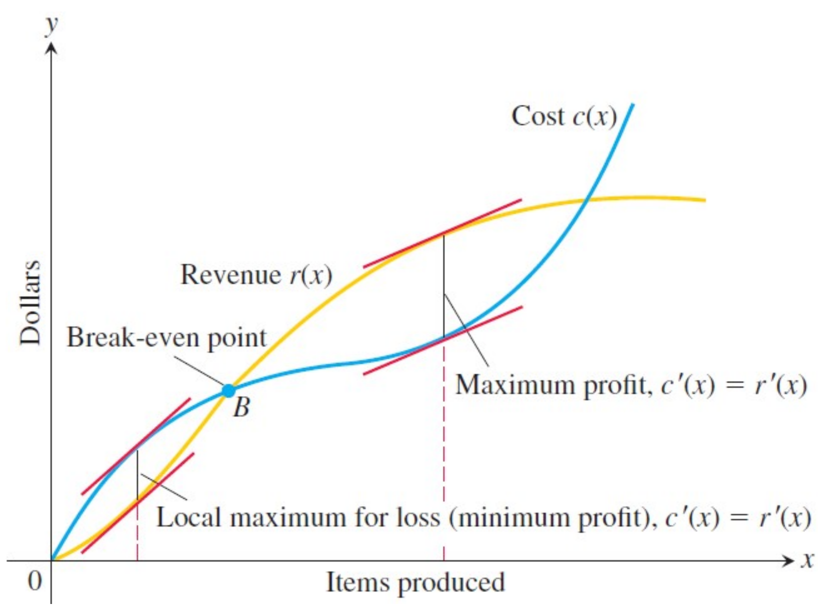

Figure 3.2: The graph of a typical cost function starts concave down and later turns concave up. It crosses the revenue curve at the break-even point B. To the left of B, the company operates at a loss. To the right, the company operates at a profit, with the maximum profit occurring where $c'(x) = r'(x)$. Farther to the right, cost exceeds revenue (perhaps because of a combination of rising labor and material costs and market saturation) and production levels become unprofitable again.

3.7 Newton's Method

The Newton method is a numerical method for finding zeros of differentiable functions. Let $f : (a, b) \to \mathbb{R}$ be a differentiable function, and $c \in (a, b)$ is a zero of f. To find an approximated value of c, the Newton method is the following iterative scheme:

1. Make an initial estimate $x_1 \in (a, b)$ that is close to c.

2. Determine a new approximation using the iterative relation:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} .$$

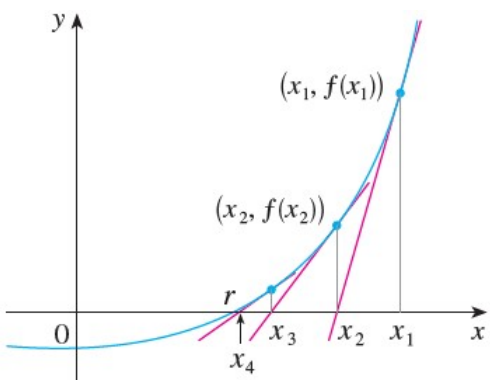

Figure 3.3: Sequence of approximated zeros by Newton's method

3. When $|x_n - x_{n+1}|$ is within the desired accuracy, let x_{n+1} serve as the final approximation.

Example 3.38. To find the square root of a positive number A is equivalent to finding zeros of the function $f(x) = x^2 - A$ in $(0, \infty)$. The Newton method provides the iterative scheme

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} = x_n - \frac{x_n^2 - A}{2x_n} = \frac{x_n}{2} + \frac{A}{2x_n}$$

to find approximated value of $\sqrt{A}$.

It can be shown that when $\left|\frac{f(x)f''(x)}{f'(x)^2}\right| < 1$ for all $x \in (a, b)$, then the Newton method produces a convergent sequence which approaches a zero in (a, b).