Calculus MA1001-B Final Exam

National Central University, Jan. 10, 2019

Problem 1. For positive integer n, let $H_n(x) = (-1)^n \exp(x^2) \frac{d^n}{dx^n} \exp(-x^2)$. Complete the following.

1. (5%) Show that H_n is a polynomial of degree n.
2. (5%) Show that $\lim_{x\to\pm\infty} p(x) \frac{d^n}{dx^n} \exp(-x^2) = 0$ for all polynomial p and positive integers n.
3. (10%) Show that $\int_{-\infty}^{\infty} \exp(-x^2) H_n(x) H_m(x)\, dx = 0$ if $m \neq n$.

Hint of 1: Prove by induction.
Hint of 2: You can use the fact that $\lim_{x\to\pm\infty} x^m e^{-x^2} = 0$.
Hint of 3: Prove first that $\int_{-\infty}^{\infty} p(x) \frac{d^n}{dx^n} e^{-x^2}\, dx = -\int_{-\infty}^{\infty} p'(x) \frac{d^{n-1}}{dx^{n-1}} e^{-x^2}\, dx$ for each polynomial p.

Proof. 1. We prove by induction.

(a) For the case $n = 1$:

$$H_1(x) = -e^{x^2} \frac{d}{dx} e^{-x^2} = -e^{x^2} e^{-x^2} (-2x) = 2x$$

which is a polynomial of degreen 1.

(b) Assume that H_k is a polynomial of degree k. Then by the fact that

$$\frac{d^k}{dx^k} e^{-x^2} = (-1)^k e^{-x^2} H_k(x)\,, \tag{$\star$}$$

we find that

$$\begin{aligned} H_{k+1}(x) &= (-1)^{k+1} e^{x^2} \frac{d}{dx} \frac{d^k}{dx^k} e^{-x^2} = (-1)^{k+1} e^{x^2} \frac{d}{dx} \Big[(-1)^k e^{-x^2} H_k(x) \Big] \\ &= -e^{x^2} \frac{d}{dx} \Big[e^{-x^2} H_k(x) \Big] = -e^{x^2} \Big[-2x e^{-x^2} H_k(x) + e^{-x^2} H_k'(x) \Big] \\ &= 2x H_k(x) - H_k'(x)\,. \end{aligned}$$

Since H_k is a polynomial of degree k, the function $y = 2x H_k(x) - H_k'(x)$ is a polynomial of degree $k + 1$.

By induction, H_n is a polynomial of degree n.

2. Using ($\star$),

$$x^m \frac{d^n}{dx^n} \exp(-x^2) = (-1)^n e^{-x^2} H_n(x) x^m\,.$$

Since $\lim_{x\to\pm\infty} x^k e^{-x^2} = 0$ for all $k \geqslant 0$, we find that $\lim_{x\to\pm\infty} e^{-x^2} p(x) = 0$ for all polynomial p; thus

$$\lim_{x\to\pm\infty} x^m \frac{d^n}{dx^n} \exp(-x^2) = (-1)^n \lim_{x\to\pm\infty} e^{-x^2} H_n(x) x^m = 0\,.$$

3. W.L.O.G. we assume that $m < n$. We first prove that for each polynomial p and $n \geqslant 1$,

$$\int_{-\infty}^{\infty} p(x)\frac{d^n}{dx^{n-1}}e^{-x^2}\,dx = -\int_{-\infty}^{\infty} p'(x)\frac{d^{n-1}}{dx^{n-1}}e^{-x^2}\,dx\,. \tag{$\star\star$}$$

To see ($\star\star$), we integrate by parts $\big($with $u = p(x)$ and $v = \dfrac{d^{n-1}}{dx^{n-1}}e^{-x^2}\big)$ and find that

$$\int p(x)\frac{d^n}{dx^{n-1}}e^{-x^2}\,dx = p(x)\frac{d^{n-1}}{dx^{n-1}}e^{-x^2} - \int p'(x)\frac{d^{n-1}}{dx^{n-1}}e^{-x^2}\,dx\,;$$

thus by the fact that $\displaystyle\lim_{x\to\pm\infty} p(x)\frac{d^n}{dx^n}\exp(-x^2) = 0$ for all polynomial p,

$$\begin{aligned}
\int_{-\infty}^{\infty} p(x)\frac{d^n}{dx^{n-1}}e^{-x^2}\,dx &= \int_{-\infty}^{0} p(x)\frac{d^n}{dx^{n-1}}e^{-x^2}\,dx + \int_{0}^{\infty} p(x)\frac{d^n}{dx^{n-1}}e^{-x^2}\,dx \\
&= \lim_{a\to-\infty}\int_{a}^{0} p(x)\frac{d^n}{dx^{n-1}}e^{-x^2}\,dx + \lim_{b\to\infty}\int_{0}^{b} p(x)\frac{d^n}{dx^{n-1}}e^{-x^2}\,dx \\
&= \lim_{a\to-\infty}\Big[p(x)\frac{d^{n-1}}{dx^{n-1}}e^{-x^2}\Big|_{x=a}^{x=0} - \int_{a}^{0} p'(x)\frac{d^{n-1}}{dx^{n-1}}e^{-x^2}\,dx\Big] \\
&\quad + \lim_{b\to\infty}\Big[p(x)\frac{d^{n-1}}{dx^{n-1}}e^{-x^2}\Big|_{x=0}^{x=b} - \int_{0}^{b} p'(x)\frac{d^{n-1}}{dx^{n-1}}e^{-x^2}\,dx\Big] \\
&= -\int_{-\infty}^{0} p'(x)\frac{d^{n-1}}{dx^{n-1}}e^{-x^2}\,dx - \int_{0}^{\infty} p'(x)\frac{d^{n-1}}{dx^{n-1}}e^{-x^2}\,dx
\end{aligned}$$

which concludes ($\star\star$). Then using ($\star$),

$$\int_{-\infty}^{\infty} \exp(-x^2)H_n(x)H_m(x)\,dx = (-1)^n\int_{-\infty}^{\infty} H_m(x)\frac{d^n}{dx^n}e^{-x^2}\,dx\,;$$

thus ($\star\star$) implies that

$$\begin{aligned}
&\int_{-\infty}^{\infty} \exp(-x^2)H_n(x)H_m(x)\,dx \\
&\quad = (-1)^n\int_{-\infty}^{\infty} H_m(x)\frac{d^n}{dx^n}e^{-x^2}\,dx = (-1)^{n+1}\int_{-\infty}^{\infty} H'_m(x)\frac{d^{n-1}}{dx^{n-1}}e^{-x^2}\,dx \\
&\quad = (-1)^n\int_{-\infty}^{\infty} H''_m(x)\frac{d^{n-2}}{dx^{n-2}}e^{-x^2}\,dx = \cdots\cdots = \int_{-\infty}^{\infty} H_m^{(n)}(x)e^{-x^2}\,dx\,.
\end{aligned}$$

Since $m < n$ and H_m is a polynomial of degree n, we must have $H_m^{(n)}(x) = 0$; thus

$$\int_{-\infty}^{\infty} \exp(-x^2)H_n(x)H_m(x)\,dx = \int_{-\infty}^{\infty} 0\cdot e^{-x^2}\,dx = 0\,. \qquad \square$$

Problem 2. Compute the indefinite integral $\displaystyle\int \tan^3 x\,dx$ using the following methods:

1. (5%) Obtain a recurrence relation for the integral of $\displaystyle\int \tan^n x\,dx$ for $n \geqslant 2$ using integration by parts and find the integral.

2. (15%) Use the substitution $t = \sec x + \tan x$ (without any other substitution of variables) and the technique of partial fractions to find the integral.

Hint for 2: When making the substitution of variable, show that

$$\int \tan^3 x\,dx = \int \Big[At + 0 + \frac{B}{t} + \frac{0}{t^2} + \frac{C}{t^3} + \frac{Dt+0}{t^2+1}\Big]\,dt\,.$$

Solution. 1. Let $u = \tan^{n-2} x$ and $v = \tan x$ (so that $dv = \sec^2\,dx = (\tan^2 x + 1)\,dx$). Then

$$\begin{aligned}\int \tan^n x\,dx &= \int \tan^{n-2}\sec^2 x\,dx - \int \tan^{n-2} x\,dx = \int \tan^{n-2} x\,d(\tan x) - \int \tan^{n-2} x\,dx \\ &= \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x\,dx\,.\end{aligned}$$

Therefore,

$$\int \tan^3 x\,dx = \frac{\tan^2 x}{2} - \int \tan x\,dx = \frac{\tan^2 x}{2} - \ln|\sec x| + C\,.$$

2. Let $t = \sec x + \tan x$. Then $\dfrac{1}{t} = \sec x - \tan x$ which implies that

$$\sec x = \frac{t^2+1}{2t} \quad \text{and} \quad \tan x = \frac{t^2-1}{2t}\,.$$

Moreover, $dt = \sec x(\tan x + \sec x)\,dx$ or equivalently, $dx = \dfrac{2dt}{t^2+1}$. Therefore,

$$\begin{aligned}\int \tan^3 x\,dx &= \int \Big(\frac{t^2-1}{2t}\Big)^3 \frac{dt}{t^2+1} = \frac{1}{8}\int \frac{t^6 - 3t^4 + 3t^2 - 1}{t^3(t^2+1)}\,dt \\ &= \frac{1}{8}\int \Big[t + \frac{-4t^4 + 3t^2 - 1}{t^3(t^2+1)}\Big]\,dt = \frac{1}{8}\int \Big[t - \frac{4t}{t^2+1} + \frac{3t^2+3-4}{t^3(t^2+1)}\Big]\,dt \\ &= \frac{1}{8}\int \Big[t + \frac{3}{t^3} - \frac{4t}{t^2+1} - \frac{4}{t^3(t^2+1)}\Big]\,dt\end{aligned}$$

By partial fraction,

$$\frac{1}{t^3(t^2+1)} = \frac{A_1}{t} + \frac{A_2}{t^2} + \frac{A_3}{t^3} + \frac{Bt+C}{t^2+1}\,.$$

Then $A_3 = \dfrac{1}{0^2+1} = 1$, and

$$\frac{A_1}{t} + \frac{A_2}{t^2} + \frac{Bt+C}{t^2+1} = \frac{1}{t^3(t^2+1)} - \frac{1}{t^3} = \frac{-1}{t(t^2+1)}\,;$$

thus $A_2 = 0$ and $A_1 = \dfrac{-1}{0^2+1} = -1$. Therefore,

$$\frac{Bt+C}{t^2+1} = \frac{-1}{t(t^2+1)} + \frac{1}{t} = \frac{t}{t^2+1}$$

which implies that $B = 1$ and $C = 0$. As a consequence,

$$\begin{aligned}&\frac{1}{4}\int \Big[t + \frac{3}{t^3} - \frac{4t}{t^2+1} - \frac{4}{t^3(t^2+1)}\Big]\,dt \\ &\quad = \frac{1}{4}\int \Big[t + \frac{3}{t^3} - \frac{4t}{t^2+1} - 4\Big(\frac{-1}{t} + \frac{1}{t^3} + \frac{t}{t^2+1}\Big)\Big]\,dt = \frac{1}{4}\int \Big[t + \frac{4}{t} - \frac{1}{t^3} - \frac{8t}{t^2+1}\Big]\,dt \\ &\quad = \frac{1}{4}\Big[\frac{t^2}{2} + 4\ln|t| + \frac{1}{2t^2} - 4\ln(t^2+1)\Big] + C_1 = \frac{1}{2}\Big[\frac{1}{2}\Big(t + \frac{1}{t}\Big)\Big]^2 - \ln\Big|\frac{1}{2}\Big(t + \frac{1}{t}\Big)\Big| + C\,;\end{aligned}$$

thus

$$\int \tan^3 x\,dx = \frac{\sec^2 x}{2} - \ln|\sec x| + C\,.$$

□

Problem 3. (15%) Evaluate $\int_0^1 x \arctan x \, dx$ using integration by parts with $u = x$ and $dv = \arctan x \, dx$.

Solution. First we find v so that $dv = \arctan x \, dx$. Integrating by parts,

$$\int \arctan x \, dx = x \arctan x - \int \frac{x}{1+x^2} \, dx = x \arctan x - \frac{1}{2} \ln(x^2+1) + C \, .$$

Let $v = x \arctan x - \frac{1}{2} \ln(x^2+1)$. Then

$$\begin{aligned}\int x \arctan x \, dx &= x\big[x \arctan x - \frac{1}{2}\ln(x^2+1)\big] - \int \big[x \arctan x - \frac{1}{2}\ln(x^2+1)\big] \, dx \\ &= x^2 \arctan x - \frac{x}{2}\ln(x^2+1) - \int x \arctan x \, dx + \frac{1}{2}\int \ln(x^2+1) \, dx \, .\end{aligned}$$

Therefore, rearranging terms we find that

$$\begin{aligned}\int x \arctan x \, dx &= \frac{x^2}{2} \arctan x - \frac{x}{4}\ln(x^2+1) + \frac{1}{4}\int \ln(x^2+1) \, dx \\ &= \frac{x^2}{2} \arctan x - \frac{x}{4}\ln(x^2+1) + \frac{1}{4}\Big[x\ln(x^2+1) - \int \frac{2x^2}{x^2+1} \, dx\Big] \\ &= \frac{x^2}{2} \arctan x - \frac{x}{4}\ln(x^2+1) + \frac{1}{4}\Big[x\ln(x^2+1) - 2\int \Big(1 - \frac{1}{1+x^2}\Big) \, dx\Big] \\ &= \frac{x^2}{2} \arctan x - \frac{x}{4}\ln(x^2+1) + \frac{1}{4}\Big[x\ln(x^2+1) - 2x + 2\arctan x\Big] + C \\ &= \frac{x^2+1}{2} \arctan x - \frac{x}{2} + C \, .\end{aligned}$$

Therefore,

$$\int_0^1 x \arctan x \, dx = \Big[\frac{x^2+1}{2} \arctan x - \frac{x}{2}\Big]\Big|_{x=0}^{x=1} = \arctan 1 - \frac{1}{2} = \frac{\pi}{4} - \frac{1}{2} \, .$$ □

Problem 4. (15%) Let n be a positive integer. Use the techniques of partial fractions to show that

$$\int \frac{dx}{x^n(x-1)^2} = n \ln\Big|\frac{x}{x-1}\Big| - \frac{1}{x-1} + \sum_{j=1}^{n-1} \frac{j}{j-n} x^{j-n} + C \, .$$

Remark: You can get 8pts if you complete the case $n = 5$.

Solution. First we write the integrand into the sum of partial fractions:

$$\frac{1}{x^n(x-1)^2} = \frac{A_1}{x} + \frac{A_2}{x^2} + \cdots + \frac{A_n}{x^n} + \frac{B}{x-1} + \frac{C}{(x-1)^2} \, .$$

Then $C = \frac{1}{1^n} = 1$; thus

$$\frac{A_1}{x} + \frac{A_2}{x^2} + \cdots + \frac{A_n}{x^n} + \frac{B}{x-1} = \frac{1}{x^n(x-1)^2} - \frac{1}{(x-1)^2} = \frac{1-x^n}{x^n(x-1)^2} = \frac{-1-x-x^2-\cdots-x^{n-1}}{x^n(x-1)} \, .$$

Therefore,

$$B = \frac{-1-1-1^2-\cdots-1^{n-1}}{1^n} = -n \, ;$$

hence

$$\begin{aligned}\frac{A_1}{x}+\frac{A_2}{x^2}+\cdots+\frac{A_n}{x^n} &= \frac{-1-x-x^2-\cdots-x^{n-1}}{x^n(x-1)}+\frac{n}{x-1}\\ &= \frac{-1-x-x^2-\cdots-x^{n-1}}{x^n(x-1)}+\frac{nx^n}{x^n(x-1)}\\ &= \frac{-1-x-x^2-\cdots-x^{n-1}+nx^n}{x^n(x-1)}\,.\end{aligned}$$

Since

$$\begin{aligned}-1-x-x^2-\cdots-x^{n-1}+nx^n &= \sum_{j=0}^{n-1}(x^n-x^j)=\sum_{j=0}^{n-1}x^j(x-1)(x^{n-j-1}+x^{n-j-2}+\cdots+x+1)\\ &= (x-1)\sum_{j=0}^{n-1}(x^{n-1}+x^{n-2}+\cdots+x^{j+1}+x^j)\\ &= (x-1)(1+2x+3x^2+\cdots+nx^{n-1})=(x-1)\sum_{j=0}^{n-1}(j+1)x^j\,,\end{aligned}$$

we find that

$$\frac{A_1}{x}+\frac{A_2}{x^2}+\cdots+\frac{A_n}{x^n}=\frac{\sum\limits_{j=0}^{n-1}(j+1)x^j}{x^n}=\sum_{j=0}^{n-1}(j+1)x^{j-n}\,.$$

As a consequence,

$$\frac{1}{x^n(x-1)^2}=\frac{1+2x+3x^2+\cdots+nx^{n-1}}{x^n}=\frac{n}{x}+\sum_{j=0}^{n-2}(j+1)x^{j-n}-\frac{n}{x-1}+\frac{1}{(x-1)^2}$$

which implies that

$$\int\frac{dx}{x^n(x-1)^2}=n\ln\left|\frac{x}{x-1}\right|-\frac{1}{x-1}+\sum_{j=1}^{n-1}\frac{j}{j-n}x^{j-n}+C\,.$$ □

Problem 5. (20%) Evaluate $\displaystyle\int_{-1}^{1}\arctan\frac{\sqrt{1-x^2}}{\sqrt{3}}\,dx$.

Solution 1. Let $x=\sin t$. Then $dx=\cos t\,dt$; thus

$$\begin{aligned}\int_{-1}^{1}\arctan\frac{\sqrt{1-x^2}}{\sqrt{3}}\,dx=\int_{-1}^{1}\arctan\frac{\sqrt{1-x^2}}{\sqrt{3}}\,dx=\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\cos t\arctan\frac{\cos t}{\sqrt{3}}\,dt\\ &= \sin t\arctan\frac{\cos t}{\sqrt{3}}\Big|_{t=-\frac{\pi}{2}}^{t=\frac{\pi}{2}}-\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\sin t\frac{\frac{-\sin t}{\sqrt{3}}}{1+\frac{\cos^2 t}{3}}\,dt\\ &= \sqrt{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin^2 t}{3+\cos^2 t}\,dt=\sqrt{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{\sin^2 t-4}{3+\cos^2 t}\,dt+4\sqrt{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{3+\cos^2 t}\,dt\\ &= -\sqrt{3}\pi+4\sqrt{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}}\frac{1}{3+\cos^2 t}\,dt\,.\end{aligned}$$

By the half angle formula,

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{3+\cos^2 t}\, dt = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{1}{3+\frac{1+\cos 2t}{2}}\, dt = 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dt}{7+\cos 2t}\,.$$

Let $u = \tan t$. Then $\cos 2t = \dfrac{1-u^2}{1+u^2}$ and $dt = \dfrac{du}{1+u^2}$; thus

$$\begin{aligned} 2\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{dt}{7+\cos 2t} &= 2\int_{-\infty}^{\infty} \frac{1}{7+\frac{1-u^2}{1+u^2}}\frac{du}{1+u^2} = 2\int_{-\infty}^{\infty}\frac{1}{8+6u^2}\,du = \frac{1}{3}\int_{-\infty}^{\infty}\frac{du}{\frac{4}{3}+u^2} \\ &= \frac{1}{3}\frac{\sqrt{3}}{2}\arctan\frac{\sqrt{3}u}{2}\Big|_{u=-\infty}^{u=\infty} = \frac{\pi}{2\sqrt{3}}\,. \end{aligned}$$

Therefore,

$$\int_{-1}^{1}\arctan\frac{\sqrt{1-x^2}}{\sqrt{3}}\,dx = -\sqrt{3}\pi + 4\sqrt{3}\cdot\frac{\pi}{2\sqrt{3}} = (2-\sqrt{3})\pi\,. \qquad \square$$

Solution 2. Let $u =$ and $v = x$. Integrating by parts, we find that

$$\begin{aligned} \int_{-1}^{1}\arctan\frac{\sqrt{1-x^2}}{\sqrt{3}}\,dx &= x\arctan\frac{\sqrt{1-x^2}}{\sqrt{3}}\Big|_{x=-1}^{x=1} - \int_{-1}^{1} x\frac{d}{dx}\arctan\frac{\sqrt{1-x^2}}{\sqrt{3}}\,dx \\ &= -\int_{-1}^{1}\frac{x\cdot\frac{1}{\sqrt{3}}\frac{-x}{\sqrt{1-x^2}}}{1+\frac{1-x^2}{3}}\,dx = \sqrt{3}\int_{-1}^{1}\frac{x^2-4+4}{(4-x^2)\sqrt{1-x^2}}\,dx \\ &= -\sqrt{3}\int_{-1}^{1}\frac{dx}{\sqrt{1-x^2}} + \sqrt{3}\int_{-1}^{1}\frac{4}{(4-x^2)\sqrt{1-x^2}}\,dx \\ &= -\sqrt{3}\arcsin x\Big|_{x=-1}^{x=1} + \sqrt{3}\int_{-1}^{1}\frac{4}{(4-x^2)\sqrt{1-x^2}}\,dx \\ &= -\sqrt{3}\pi + \sqrt{3}\int_{-1}^{1}\frac{4}{(4-x^2)\sqrt{1-x^2}}\,dx\,. \end{aligned}$$

Using the substitution $x = \sin\dfrac{\theta}{2}$, we find that

$$\int_{-1}^{1}\frac{4}{(4-x^2)\sqrt{1-x^2}}\,dx = \int_{-\pi}^{\pi}\frac{4}{2(4-\sin^2\frac{\theta}{2})}\,d\theta = \int_{-\pi}^{\pi}\frac{4}{8-(1-\cos\theta)}\,d\theta = \int_{-\pi}^{\pi}\frac{4}{7+\cos\theta}\,d\theta\,.$$

and further substitution $\tan\dfrac{\theta}{2} = t$ implies that

$$\begin{aligned} \int_{-1}^{1}\frac{4}{(4-x^2)\sqrt{1-x^2}}\,dx &= \int_{-\infty}^{\infty}\frac{4}{7+\frac{1-t^2}{1+t^2}}\frac{2dt}{1+t^2} = \int_{-\infty}^{\infty}\frac{8}{7(1+t^2)+(1-t^2)}\,dt \\ &= \int_{-\infty}^{\infty}\frac{4}{4+3t^2}\,dt = \frac{4}{3}\int_{-\infty}^{\infty}\frac{dt}{t^2+\left(\frac{2}{\sqrt{3}}\right)^2} \\ &= \frac{2}{\sqrt{3}}\arctan\left(\frac{\sqrt{3}}{2}t\right)\Big|_{t=-\infty}^{\infty} = \frac{2\pi}{\sqrt{3}}\,. \end{aligned}$$

Therefore,

$$\int_{-1}^{1}\arctan\frac{\sqrt{1-x^2}}{\sqrt{3}}\,dx = -\sqrt{3}\pi + \sqrt{3}\cdot\frac{2\pi}{\sqrt{3}} = (2-\sqrt{3})\pi\,. \qquad \square$$

Problem 6. (10%) Show that the improper integral $\displaystyle\int_0^1 \ln\sin x\,dx$ converges using the comparison test for improper integrals.

Solution 1. Let $f(x) = -\ln\sin x$ and $g(x) = -\ln x$. Then f, g are positive on $(0,1]$. Moreover, by L'Hôspital's rule,

$$\lim_{x\to 0^+}\frac{f(x)}{g(x)} = \lim_{x\to 0^+}\frac{\ln\sin x}{\ln x} = \lim_{x\to 0^+}\frac{\cos x/\sin x}{1/x} = \lim_{x\to 0^+}\frac{x\cos x}{\sin x} = 1 > 0\,.$$

Therefore, by the limit comparison test,

$$\int_0^1 \ln\sin x\,dx \ \text{ converges if and only if } \int_0^1 \ln x\,dx \ \textit{converges}.$$

Now, by the fact that $\displaystyle\lim_{x\to 0^+} x\ln x = 0$,

$$\int_0^1 \ln x\,dx = \lim_{a\to 0^+}\int_a^1 \ln x\,dx = \lim_{a\to 0^+}\big(x\ln x - x\big)\Big|_{x=a}^{x=1} = \lim_{a\to 0^+}(a - 1 - a\ln a) = -1\,;$$

thus $\displaystyle\int_0^1 \ln x\,dx$ converges. □

Solution 2. Note that $\frac{2}{\pi}x \leqslant \sin x \leqslant x$ for all $x \in \big(0, \frac{\pi}{2}\big)$. Since $y = \ln x$ is increasing on $(0,1)$,

$$\ln\frac{2}{\pi} + \ln x \leqslant \ln\sin x \leqslant \ln x \leqslant 0 \qquad \forall\, x \in (0,1)\,.$$

Let $f(x) = -\ln\sin x$ and $g(x) = -\ln\frac{2}{\pi} - \ln x$. Then $0 \leqslant f(x) \leqslant g(x)$ for all $x \in (0,1]$. Now, by the fact that $\displaystyle\lim_{x\to 0^+} x\ln x = 0$,

$$\int_0^1 \ln x\,dx = \lim_{a\to 0^+}\int_a^1 \ln x\,dx = \lim_{a\to 0^+}\big(x\ln x - x\big)\Big|_{x=a}^{x=1} = \lim_{a\to 0^+}(a - 1 - a\ln a) = -1\,;$$

thus $\displaystyle\int_0^1 \ln x\,dx$ converges. Therefore, $\displaystyle\int_0^1 g(x)\,dx = -\ln\frac{2}{\pi} - \int_0^1 \ln x\,dx$ converges. By the direct comparison test, $\displaystyle\int_0^1 \ln\sin x\,dx$ converges. □