Calculus MA1001-B Quiz 13

National Central University, Dec. 31 2019

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Problem 1. (5pts) Find the indefinite integral $\displaystyle\int \frac{-4x-2}{(x+1)^2(x^2+1)}\,dx$.

Solution. First by writing the integrand into the sum of partial fractions,

$$\frac{-4x-2}{(x+1)^2(x^2+1)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{x^2+1}\,.$$

The constant B is determined by $B = \dfrac{-4(-1)-2}{(-1)^2+1} = 1$; thus

$$\begin{aligned}\frac{A}{x+1} + \frac{Cx+D}{x^2+1} &= \frac{-4x-2}{(x+1)^2(x^2+1)} - \frac{1}{(x+1)^2} = \frac{-4x-2-(x^2+1)}{(x+1)^2(x^2+1)} = \frac{-x^2-4x-3}{(x+1)^2(x^2+1)} \\ &= \frac{-(x+3)(x+1)}{(x+1)^2(x^2+1)} = \frac{-x-3}{(x+1)(x^2+1)}\,.\end{aligned}$$

Having the identity above, we find that $A = \dfrac{-(-1)-3}{(-1)^2+1} = -1$. This then implies that

$$\frac{Cx+D}{x^2+1} = \frac{-x-3}{(x+1)(x^2+1)} + \frac{1}{x+1} = \frac{-x-3+x^2+1}{(x+1)(x^2+1)} = \frac{(x-2)(x+1)}{(x+1)(x^2+1)} = \frac{x-2}{x^2+1}\,.$$

Therefore,

$$\begin{aligned}\int \frac{-4x-2}{(x+1)^2(x^2+1)}\,dx &= \int \Big[\frac{-1}{x+1} + \frac{1}{(x+1)^2} + \frac{x-2}{x^2+1}\Big]\,dx \\ &= -\ln|x+1| - \frac{1}{x+1} + \frac{1}{2}\ln(x^2+1) - 2\arctan x + C\,.\end{aligned}$$

□

Problem 2. (2pts) Find the value of the improper integral $\displaystyle\int_1^\infty (1-x)e^{-x}\,dx$.

Solution. We first compute the indefinite integral $\displaystyle\int (1-x)e^{-x}\,dx$. Let $u = (1-x)$ and $v = -e^{-x}$ (so that $dv = e^{-x}\,dx$). Then integration-by-parts shows that

$$\int (1-x)e^{-x}\,dx = -(1-x)e^{-x} + \int e^{-x}(-1)\,dx = (x-1)e^{-x} + e^{-x} + C = xe^{-x} + C\,.$$

Therefore,

$$\int_1^\infty (1-x)e^{-x}\,dx = \lim_{b\to\infty}\int_1^b (1-x)e^{-x}\,dx = \lim_{b\to\infty}\Big(xe^{-x}\Big|_{x=1}^{x=b}\Big) = \lim_{b\to\infty}\big(be^{-b} - e^{-1}\big) = -e^{-1}\,. \qquad \square$$

Problem 3. (3pts) Show that the improper integral $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$ converges if and only if the improper integral $\displaystyle\int_0^\infty \frac{1-\cos x}{x^2}\,dx$ converges.

Solution. Let $u = \dfrac{1}{x}$ and $v = 1-\cos x$ (so that $dv = \sin x\,dx$). Then by the definition of the improper integrals,

$$\int_0^\infty \frac{\sin x}{x}\,dx = \int_0^{2\pi} \frac{\sin x}{x}\,dx + \int_1^\infty \frac{\sin x}{x}\,dx = \lim_{a\to 0^+}\int_a^{2\pi} \frac{\sin x}{x}\,dx + \lim_{b\to\infty}\int_{2\pi}^b \frac{\sin x}{x}\,dx\,.$$

Integrating by parts, we find that

$$\int_a^{2\pi} \frac{\sin x}{x}\,dx = \frac{1-\cos x}{x}\Big|_{x=a}^{x=2\pi} + \int_a^{2\pi} \frac{1-\cos x}{x^2}\,dx = \frac{\cos a - 1}{a} + \int_a^{2\pi} \frac{1-\cos x}{x^2}\,dx$$

and

$$\int_{2\pi}^b \frac{\sin x}{x}\,dx = \frac{1-\cos x}{x}\Big|_{x=2\pi}^{x=b} + \int_{2\pi}^b \frac{1-\cos x}{x^2}\,dx = \frac{1-\cos b}{b} + \int_{2\pi}^b \frac{1-\cos x}{x^2}\,dx\,.$$

By the fact that

$$\lim_{a\to 0^+}\frac{\cos a - 1}{a} = \lim_{b\to\infty}\frac{1-\cos b}{b} = 0\,,$$

we find that

$$\int_0^{2\pi} \frac{\sin x}{x}\,dx \;\text{ converges if and only if }\; \int_0^{2\pi} \frac{1-\cos x}{x^2}\,dx \;\text{ converges}$$

and

$$\int_{2\pi}^\infty \frac{\sin x}{x}\,dx \;\text{ converges if and only if }\; \int_{2\pi}^\infty \frac{1-\cos x}{x^2}\,dx \;\text{ converges.}$$

Therefore,

$$\begin{aligned}
\int_0^\infty \frac{\sin x}{x}\,dx \;\text{ converges} &\Leftrightarrow \int_0^{2\pi} \frac{\sin x}{x}\,dx \;\text{ converges and }\; \int_{2\pi}^\infty \frac{\sin x}{x}\,dx \;\text{ converges} \\
&\Leftrightarrow \int_0^{2\pi} \frac{1-\cos x}{x^2}\,dx \;\text{ converges and }\; \int_{2\pi}^\infty \frac{\sin x}{x}\,dx \;\text{ converges} \\
&\Leftrightarrow \int_0^\infty \frac{1-\cos x}{x^2}\,dx \;\text{ converges.}
\end{aligned} \qquad \square$$