Calculus MA1001-B Quiz 10

National Central University, Dec. 03 2019

學號：____________________ 姓名：____________________

Problem 1. (2pts) Use the definition of exp to show that $\exp(a+b) = \exp(a)\exp(b)$ for all $a, b \in \mathbb{R}$.

Proof. Let $a, b \in \mathbb{R}$ be given, and $\exp(a) = c$, $\exp(b) = d$. By definition, exp is the inverse function of ln; thus $a = \ln c$ and $b = \ln d$. By the logarithmic property of ln, $\ln(cd) = \ln c + \ln d$; thus

$$\exp(a+b) = \exp(\ln c + \ln d) = \exp(\ln(cd)) = cd = \exp(a)\exp(b)\,. \qquad \square$$

Problem 2. (4pts) Show that the function $f(x) = \big(1 + \frac{1}{x}\big)^{x+1}$ is decreasing on $(0, \infty)$.

Solution. Note that $f(x) = \exp\big((x+1)\ln(1+\frac{1}{x})\big)$. Therefore,

$$\begin{aligned} f'(x) &= \exp\big((x+1)\ln(1+\frac{1}{x})\big)\frac{d}{dx}\Big[(x+1)\ln(1+\frac{1}{x})\Big] = f(x)\Big[\ln(1+\frac{1}{x}) + (x+1)\frac{-1/x^2}{1+1/x}\Big] \\ &= f(x)\big[\ln\big(1+\frac{1}{x}\big) - \frac{1}{x}\big]\,. \end{aligned}$$

Let $g(x) = \ln\big(1+\frac{1}{x}\big) - \frac{1}{x}$. Then

$$g'(x) = \frac{-1/x^2}{1+1/x} + \frac{1}{x^2} = -\frac{1}{x(x+1)} + \frac{1}{x^2} = \frac{-x+(x+1)}{x^2(x+1)} = \frac{1}{x^2(x+1)} > 0\,;$$

thus g is increasing on $(0, \infty)$ and for $x > 0$, $g(x) \leqslant \lim\limits_{x\to\infty} g(x) = 0$. Therefore, g is non-positive on $(0, \infty)$ which implies that $f'(x) \leqslant 0$ for all $x > 0$. As a consequence, f is decreasing on $(0, \infty)$. $\square$

Problem 3. (4pts) Find all the asymptotes of the graph of the function $f(x) = \dfrac{2^x + 5^{-x}}{5\cdot 2^x - 2\cdot 5^{-x}}$.

Solution. First we note that the denominator has a zero at $c = 2\log_{10} 2 - 1$ since

$$5\cdot 2^x = 2\cdot 5^{-x} \quad \Leftrightarrow \quad 10^x = 0.4 \quad \Leftrightarrow \quad x = \log_{10} 0.4 = 2\log_{10} 2 - 1\,.$$

Moreover, $\lim\limits_{x\to c^+} f(x) = \infty$ and $\lim\limits_{x\to c^-} f(x) = -\infty$, we find that $x = c$ is a vertical asymptote.

Next we check if there are slant/horizontal asymptote. Note that

$$\lim_{x\to\infty}\frac{f(x)}{x} = \lim_{x\to\infty}\frac{2^x + 5^{-x}}{x(5\cdot 2^x - 2\cdot 5^{-x})} = \lim_{x\to\infty}\frac{1 + 10^{-x}}{x(5 - 2\cdot 10^{-x})} = 0$$

and

$$\lim_{x\to-\infty}\frac{f(x)}{x} = \lim_{x\to-\infty}\frac{2^x + 5^{-x}}{x(5\cdot 2^x - 2\cdot 5^{-x})} = \lim_{x\to-\infty}\frac{10^x + 1}{x(5\cdot 10^x - 2)} = 0\,.$$

Moreover,

$$\lim_{x\to\infty}\big[f(x) - 0\cdot x\big] = \lim_{x\to\infty}\frac{2^x + 5^{-x}}{5\cdot 2^x - 2\cdot 5^{-x}} = \lim_{x\to\infty}\frac{1 + 10^{-x}}{5 - 2\cdot 10^{-x}} = \frac{1}{5}$$

and

$$\lim_{x\to-\infty}\big[f(x) - 0\cdot x\big] = \lim_{x\to-\infty}\frac{2^x + 5^{-x}}{5\cdot 2^x - 2\cdot 5^{-x}} = \lim_{x\to-\infty}\frac{10^x + 1}{5\cdot 10^x - 2} = -\frac{1}{2}\,.$$

Therefore, $y = \frac{1}{5}$ and $y = -\frac{1}{2}$ are horizontal asymptotes of the graph of f. $\square$