Calculus MA1001-B Quiz 6

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Problem 1. (4pts) Let $f : [a, b] \to \mathbb{R}$ be a function. Assuming that you know what a partition (but nothing else) of an interval is, state the definition of the integrability of f on $[a, b]$. (假設只有「一個區間的分割」這件事是已定義的情況下，敘述函數 f 在 $[a, b]$ 上黎曼可積的定義)

Solution. f is said to be Riemann integrable on $[a, b]$ if there exists a real number A such that for every $\varepsilon > 0$ there exists $\delta > 0$ such that if $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_n = b\}$ is a partition of $[a, b]$ satisfying that $\max\{x_i - x_{i-1} \mid 1 \leqslant i \leqslant n\} < \delta$ and $\{c_1, \cdots, c_n\}$ satisfying that $c_i \in [x_{i-1}, x_i]$ for all $1 \leqslant i \leqslant n$, then

$$A - \varepsilon < \sum_{i=1}^{n} f(c_i)(x_i - x_{i-1}) < A + \varepsilon\,.$$

□

Problem 2. (2pts) Write the limit $\lim\limits_{n\to\infty} \sum\limits_{i=1}^{n} \frac{1}{3n + 2i}$ as an integral. Do **NOT** compute the integral.

Solution. Let $\mathcal{P} = \{0 = x_0 < x_1 < \cdots < x_n = 1\}$ be a regular partition of $[0, 1]$. Then

$$\sum_{i=1}^{n} \frac{1}{3n + 2i} = \sum_{i=1}^{n} \frac{1}{3 + 2 \cdot \frac{i}{n}} \frac{1}{n} = \sum_{i=1}^{n} f\big(\frac{i}{n}\big)(x_i - x_{i-1})$$

which is a Riemann of f for $\mathcal{P}$ (using the right end-point rule), where $f(x) = \frac{1}{3 + 2x}$. Therefore,

$$\lim_{n\to\infty} \sum_{i=1}^{n} \frac{1}{3n + 2i} = \int_0^1 \frac{1}{3 + 2x}\,dx\,.$$

□

Problem 3. (4pts) Find the area of the region enclosed by the graph of $y = x^3$, the x-axis, and $x = 2$, using regular partitions of $[0, 2]$ and the **mid-point rule** approximation of the area.

Solution. Let $f(x) = x^3$ and $\mathcal{P} = \{0 = x_0 < x_1 < \cdots < x_n = 2\}$ be a regular partition of $[0, 2]$, where $x_i = \frac{2i}{n}$. Using the mid-point rule, we find that the area of the region given above is the limit, as $n \to \infty$, of the sum

$$\sum_{i=1}^{n} f\big(\frac{x_i + x_{i-1}}{2}\big)(x_i - x_{i-1}) = \sum_{i=1}^{n} \big(\frac{2i + 2(i-1)}{2n}\big)^3 \frac{2}{n} = \frac{2}{n^4} \sum_{i=1}^{n} (8i^3 - 12i^2 + 6i - 1)\,.$$

By the fact that $\sum\limits_{i=1}^{n} 1 = n$, $\sum\limits_{i=1}^{n} i = \frac{n(n+1)}{2}$, $\sum\limits_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}$ and $\sum\limits_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}$, we find that

$$\sum_{i=1}^{n} f\big(\frac{x_i + x_{i-1}}{2}\big)(x_i - x_{i-1}) = \frac{2}{n^4}\Big[2n^2(n+1)^2 - 2n(n+1)(2n+1) + 3n(n+1) - n\Big]\,.$$

Therefore, using that $\lim\limits_{n\to\infty} \frac{1}{n} = 0$, we conclude that

$$\lim_{n\to\infty} \sum_{i=1}^{n} f\big(\frac{x_i + x_{i-1}}{2}\big)(x_i - x_{i-1}) = \lim_{n\to\infty} \Big[4\big(1 + \frac{1}{n}\big)^2 - \frac{4}{n}\big(1 + \frac{1}{n}\big)\big(2 + \frac{1}{n}\big) + \frac{6}{n^2}\big(1 + \frac{1}{n}\big) - \frac{2}{n^3}\Big] = 4\,.$$

□