Calculus MA1001-B Quiz 5

National Central University, Oct. 22 2019

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Problem 1. (2pts) State the Mean Value Theorem.

Solution. Let $f : [a, b] \to \mathbb{R}$ be continuous. If f is differentiable on (a, b), then there exists $c \in (a, b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}\,. \qquad \square$$

Problem 2. Let $f : [0, \pi] \to \mathbb{R}$ be defined by $f(x) = \dfrac{x^2}{\pi} + \cos x$, and $x_0 = \dfrac{\pi}{2}$.

(a) (1pt) Show that x_0 is a critical point of f.

(b) (1pt) Use the second derivative test to determine whether $f(x_0)$ is a local maximum or local minimum of f.

(c) (3pt) Show that f is strictly decreasing on $(0, x_0)$ and strictly increasing on (x_0, π).

(d) (1pts) Use the first derivative test to determine whether $f(x_0)$ is a local maximum or local minimum of f.

(e) (2pts) Show that $f(x) \geqslant 1$ for all $x \in [0, \pi]$.

Solution. (a) Since $f'(x) = \dfrac{2x}{\pi} - \sin x$, $f'(x_0) = \dfrac{2}{\pi} \cdot \dfrac{\pi}{2} - \sin \dfrac{\pi}{2} = 0$; thus x_0 is a critical point of f.

(b) Since $f''(x_0) = \dfrac{2}{\pi} - \cos x_0 = \dfrac{2}{\pi} > 0$, the second derivative test implies that $f(x_0)$ is a local minimum of f.

(c) Note that $f'(x) = \dfrac{2x}{\pi} - \sin x$. Clearly, $f'(x) > 0$ if $x \in (x_0, \pi)$; thus f is strictly increasing on (x_0, π). Next we show that $f'(x) < 0$ for all $x \in (0, x_0)$. To see this, we let $g(x) = f'(x)$. Note that $g'(x) = \dfrac{2}{\pi} - \cos x$. Since the cosine function is strictly decreasing on $[0, \pi]$, g' is strictly increasing on $[0, \pi]$. By the fact that $\cos \pi = -1 < \dfrac{2}{\pi} < 1 = \cos 0$, the Intermediate Value Theorem implies that there is only one $x_0 \in [0, \pi]$ satisfying that $g'(x_0) = 0$. Since $g''(x_0) = \sin x_0 > 0$, $g(x_0)$ is a local minimum of g; thus g attains its maximum at 0 or x_0. Since $g(0) = 0$ and $g(x_0) = 0$, we find that $g(x) < 0$ for all $x \in (0, x_0)$.

(d) From (c), we find that f' changes from negative to positive at x_0; thus $f(x_0)$ is a local minimum of f.

(e) From (c) and the fact that x_0 is a critical point of f, we find that x_0 is the only critical point of f. Now, $f(x_0) = \dfrac{\pi}{4}$, $f(0) = 1$ and $f(\pi) = \pi - 1 > 1$, we find that the minimum of f on $[0, \pi]$ is $f(x_0) = \dfrac{\pi}{4}$; thus $f(x) \geqslant \dfrac{\pi}{4}$ all $x \in [0, \pi]$. $\square$