Calculus MA1001-B Quiz 3

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Problem 1. (2%) Let f be a function defined on an open interval I, and $c \in I$. State the definition of the differentiability (可微性) of f at c.

Solution. f is said to be differentiable at c if the limit $\displaystyle\lim_{h\to 0}\frac{f(c+h)-f(c)}{h}$ exists. □

Problem 2. (3%) Use the definition of differentiability of a function to find the derivative of $y = x^{\frac{1}{3}}$.

Solution. For $x \neq 0$ and $h \neq 0$,

$$\frac{(x+h)^{\frac{1}{3}} - x^{\frac{1}{3}}}{h} = \frac{(x+h)^{\frac{3}{3}} - x^{\frac{3}{3}}}{h\big[(x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}\big]} = \frac{1}{(x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}}\,.$$

Therefore, if $x \neq 0$,

$$\lim_{h\to 0}\frac{(x+h)^{\frac{1}{3}} - x^{\frac{1}{3}}}{h} = \lim_{h\to 0}\frac{1}{(x+h)^{\frac{2}{3}} + (x+h)^{\frac{1}{3}}x^{\frac{1}{3}} + x^{\frac{2}{3}}} = \frac{1}{3}x^{-\frac{2}{3}}\,.$$

On the other hand, $\displaystyle\lim_{h\to 0}\frac{(0+h)^{\frac{1}{3}} - 0^{\frac{1}{3}}}{h} = \lim_{h\to 0} h^{-\frac{2}{3}}$ does not exist, we conclude that

$$\frac{d}{dx}\Big|_{x=c} x^{\frac{1}{3}} = \begin{cases} \frac{1}{3}c^{-\frac{2}{3}} & \text{if } c \neq 0\,, \\ \text{D.N.E.} & \text{if } c = 0\,. \end{cases}$$

□

Problem 3. (5%) Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function, and

$$f(\tan x) = \frac{\sin(4x)}{x^2+1} \qquad \forall\, x \in \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big)\,.$$

Find $f'(1)$.

Solution. Since f is differentiable on $\mathbb{R}$ and tan is differentiable on $\big(-\frac{\pi}{2}, \frac{\pi}{2}\big)$, by the chain rule we find that the function $y = f(\tan x)$ is differentiable on $\big(-\frac{\pi}{2}, \frac{\pi}{2}\big)$, and

$$\frac{d}{dx}f(\tan x) = f'(\tan x)\sec^2 x \qquad \forall\, x \in \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big)\,.$$

On the other hand, since $f(\tan x) = \dfrac{\sin(4x)}{x^2+1}$ and

$$\frac{d}{dx}\frac{\sin(4x)}{x^2+1} = \frac{4(x^2+1)\cos(4x) - 2x\sin(4x)}{(x^2+1)^2}\,,$$

we find that

$$f'(\tan x)\sec^2 x = \frac{4(x^2+1)\cos(4x) - 2x\sin(4x)}{(x^2+1)^2} \qquad \forall\, x \in \Big(-\frac{\pi}{2}, \frac{\pi}{2}\Big)\,.$$

In particular, for $x = \frac{\pi}{4}$, we find that

$$f'(1)\cdot(\sqrt{2})^2 = \frac{4\big(\frac{\pi^2}{16}+1\big)\cos\pi - 2\cdot\frac{\pi}{4}\sin\pi}{\big(\frac{\pi^2}{16}+1\big)^2} = \frac{-64}{\pi^2+16}\,;$$

thus $f'(1) = \dfrac{-32}{\pi^2+16}$. □