Calculus MA1001-B Quiz 2

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Problem 1. (2%) Let f be a function defined on $(-\infty, a)$. State the definition of $\lim\limits_{x\to-\infty} f(x) = \infty$. Do **NOT** use logic symbols.

Solution. $\lim\limits_{x\to-\infty} f(x) = \infty$ if for every $M > 0$, there exists $N > 0$ such that $f(x) > M$ whenever $x < -N$. □

Problem 2. (8%) Find all asymptotes of the graph of the function $f(x) = \dfrac{3x^3(x - \sqrt[3]{x^3 - x^2 + x})}{x^2 - 1}$.

Solution. Since the denominator vanishes at $x = \pm 1$, there are two possible vertical asymptotes $x = 1$ or $x = -1$. Since the denominator also vanishes at $x = 1$, we need to check further the behavior of $f(x)$ as x approaches 1. Note that for $x \neq \pm 1$,

$$\frac{x - \sqrt[3]{x^3 - x^2 + x}}{x^2 - 1} = \frac{x}{(x+1)\big[x^2 + x\sqrt[3]{x^3 - x^2 + x} + (x^3 - x^2 + x)^{\frac{2}{3}}\big]};$$

thus for $x \neq \pm 1$,

$$f(x) = \frac{3x^4}{(x+1)\big[x^2 + x\sqrt[3]{x^3 - x^2 + x} + (x^3 - x^2 + x)^{\frac{2}{3}}\big]}.$$

Therefore, $\lim\limits_{x\to 1} f(x) = 0$ exists which shows that $x = 1$ is not a vertical asymptote of the graph of f. On the other hand,

$$\lim_{x\to -1^+} f(x) = \infty \qquad \text{and} \qquad \lim_{x\to -1^-} f(x) = -\infty\,,$$

we find that $\underline{\underline{x = -1 \text{ is the only vertical asymptote of the graph of } f}}$.

For slant or horizontal asymptotes, we note that for $x \neq \pm 1, 0$,

$$\frac{f(x)}{x} = \frac{3}{(1+\frac{1}{x})\big[1 + \big(1 - \frac{1}{x} + \frac{1}{x^2}\big)^{\frac{1}{3}} + \big(1 - \frac{1}{x} + \frac{1}{x^2}\big)^{\frac{2}{3}}\big]}. \qquad (\star)$$

Since $\lim\limits_{x\to\pm\infty} \dfrac{1}{x} = 0$, we find that $\lim\limits_{x\to\infty} \dfrac{f(x)}{x} = 1$ and $\lim\limits_{x\to-\infty} \dfrac{f(x)}{x} = 1$. It remains to find the limit $\lim\limits_{x\to\infty} \big[f(x) - x\big]$ and $\lim\limits_{x\to-\infty} \big[f(x) - x\big]$. Using $(\star)$,

$$f(x) - x = \frac{3x - (x+1)\big[1 + \big(1 - \frac{1}{x} + \frac{1}{x^2}\big)^{\frac{1}{3}} + \big(1 - \frac{1}{x} + \frac{1}{x^2}\big)^{\frac{2}{3}}\big]}{(1+\frac{1}{x})\big[1 + \big(1 - \frac{1}{x} + \frac{1}{x^2}\big)^{\frac{1}{3}} + \big(1 - \frac{1}{x} + \frac{1}{x^2}\big)^{\frac{2}{3}}\big]}.$$

Noting that the denominator approaches 3 as x approaches $\pm\infty$, we only focus on the limit of the numerator. Since

$$\begin{aligned}
&3x - (x+1)\Big[1 + \big(1 - \frac{1}{x} + \frac{1}{x^2}\big)^{\frac{1}{3}} + \big(1 - \frac{1}{x} + \frac{1}{x^2}\big)^{\frac{2}{3}}\Big] \\
&\quad = 3x - (x+1)\Big[3 + \Big(\big(1 - \frac{1}{x} + \frac{1}{x^2}\big)^{\frac{1}{3}} - 1\Big) + \Big(\big(1 - \frac{1}{x} + \frac{1}{x^2}\big)^{\frac{2}{3}} - 1\Big)\Big] \\
&\quad = -3 - \Big[\big(1 - \frac{1}{x} + \frac{1}{x^2}\big)^{\frac{1}{3}} - 1\Big]\Big[\big(1 - \frac{1}{x} + \frac{1}{x^2}\big)^{\frac{1}{3}} + 2\Big] - x\Big[\big(1 - \frac{1}{x} + \frac{1}{x^2}\big)^{\frac{1}{3}} - 1\Big]\Big[\big(1 - \frac{1}{x} + \frac{1}{x^2}\big)^{\frac{1}{3}} + 2\Big],
\end{aligned}$$

to find the limit of the numerator as $x \to \pm\infty$ it suffices to find the limit

$$\lim_{x\to\infty} x\Big[\big(1-\frac{1}{x}+\frac{1}{x^2}\big)^{\frac{1}{3}} - 1\Big] \qquad \text{and} \qquad \lim_{x\to-\infty} x\Big[\big(1-\frac{1}{x}+\frac{1}{x^2}\big)^{\frac{1}{3}} - 1\Big]\,.$$

Now,

$$\begin{aligned}\lim_{x\to\infty} x\Big[\big(1-\frac{1}{x}+\frac{1}{x^2}\big)^{\frac{1}{3}} - 1\Big] &= \lim_{x\to 0^+} \frac{\big(1-x+x^2\big)^{\frac{1}{3}} - 1}{x} = \lim_{x\to 0^+} \frac{(1-x+x^2) - 1^3}{x\big[\big(1-x+x^2\big)^{\frac{2}{3}} + \big(1-x+x^2\big)^{\frac{1}{3}} + 1\big]} \\ &= \lim_{x\to 0^+} \frac{x-1}{\big(1-x+x^2\big)^{\frac{2}{3}} + \big(1-x+x^2\big)^{\frac{1}{3}} + 1} = -\frac{1}{3}\end{aligned}$$

and similarly, $\lim\limits_{x\to-\infty} x\Big[\big(1-\frac{1}{x}+\frac{1}{x^2}\big)^{\frac{1}{3}} - 1\Big] = -\dfrac{1}{3}$. Therefore,

$$\lim_{x\to\pm\infty} \Big[3x - (x+1)\big[1-\frac{1}{x}+\frac{1}{x^2}\big)^{\frac{1}{3}} + \big(1-\frac{1}{x}+\frac{1}{x^2}\big)^{\frac{2}{3}}\big]\Big] = -3 + \frac{1}{3}\cdot 3 = -2\,;$$

thus $\lim\limits_{x\to\infty} \big[f(x)-x\big] = -\dfrac{2}{3}$ which implies that $y = x - \dfrac{2}{3}$ is the only slant asymptote of the graph of f.

□