Calculus MA1001-A Midterm 1

National Central University, Oct. 29, 2019

Problem 1. Complete the following.

1. (5pts) Let I be an interval, and $f : I \to \mathbb{R}$ be a function. State the ε-δ definition of the continuity of f at a point $c \in I$. (敘述 f 在 I 中一點 c 連續的 ε-δ 定義)

2. (5pts) State Rolle's Theorem.

Problem 2. (10pts) State and prove the Mean Value Theorem.

Problem 3. 1. (15pts) Show that

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} \leqslant \sin x \leqslant x - \frac{x^3}{3!} \qquad \text{for all} \quad x \leqslant 0\,. \tag{$\star$}$$

Note that you have to start with the well-known inequality $\sin x \leqslant x$ for all $x \geqslant 0$.

2. (5pts) Find the limit $\lim\limits_{x\to 0} \dfrac{\sin x - x}{x^3}$. (Do not use L'Hôspital's rule even if you know this).

Solution. 1. First, we note that using the inequality $\sin x \leqslant x$ for all $x \geqslant 0$, we have "$\sin(-x) \leqslant (-x)$ for all $x \leqslant 0$"; thus

$$x \leqslant \sin x \qquad \forall\, x \leqslant 0\,. \tag{1}$$

Let $g_1(x) = \cos x - 1 + \dfrac{x^2}{2!}$. Then $g_1'(x) = -\sin x + x$; thus using (1), we find that $g_1'(x) \leqslant 0$ for all $x \leqslant 0$. Therefore, g_1 is decreasing on $(-\infty, 0]$ which shows that

$$g_1(x) \geqslant g_1(0) = 0 \qquad \forall\, x \leqslant 0\,. \tag{2}$$

Let $g_2(x) = \sin x - x + \dfrac{x^3}{3!}$. Then $g_2'(x) = \cos x - 1 + \dfrac{x^2}{2!} = g_1(x)$; thus using (2) we find that $g_2'(x) \geqslant 0$ for all $x \leqslant 0$. Therefore, g_2 is increasing on $(-\infty, 0]$ which shows that

$$g_2(x) \leqslant g_2(0) = 0 \qquad \forall\, x \leqslant 0\,. \tag{3}$$

Let $g_3(x) = \cos x - 1 + \dfrac{x^2}{2!} - \dfrac{x^4}{4!}$. Then $g_3'(x) = -\sin x + x - \dfrac{x^3}{3!} = -g_2(x)$; thus using (3) we find that $g_3'(x) \geqslant 0$ for all $x \leqslant 0$. Therefore, g_3 is increasing on $(-\infty, 0]$ which shows that

$$g_3(x) \leqslant g_3(0) = 0 \qquad \forall\, x \leqslant 0\,. \tag{4}$$

Let $g_4(x) = \sin x - x + \dfrac{x^3}{3!} - \dfrac{x^5}{5!}$. Then $g_4'(x) = \cos x - 1 + \dfrac{x^2}{2!} - \dfrac{x^4}{4!}$; thus using (4) we find that $g_4'(x) \leqslant 0$ for all $x \leqslant 0$. Therefore, g_4 is decreasing on $(-\infty, 0]$ which shows that

$$g_4(x) \geqslant g_4(0) = 0 \qquad \forall\, x \leqslant 0\,. \tag{5}$$

The desired inequality is the combination of (3) and (5).

2. Using the inequality in 1, we have

$$-\frac{1}{6} \leqslant \frac{\sin x - x}{x^3} \leqslant -\frac{1}{6} + \frac{x^2}{120} \qquad \forall\, x < 0\,.$$

Since the limits of the left-hand side and the right-hand side as $x \to 0^-$ is 0, by the Squeeze Theorem we find that

$$\lim_{x\to 0^-} \frac{\sin x - x}{x^3} = -\frac{1}{6}\,.$$

Therefore,

$$\lim_{x\to 0^+} \frac{\sin x - x}{x^3} = \lim_{x\to 0^-} \frac{\sin(-x) - (-x)}{(-x)^3} = \lim_{x\to 0^-} \frac{\sin x - x}{x^3} = -\frac{1}{6}$$

and the fact that the left-hand limit and the right-hand limit are identical implies that

$$\lim_{x\to 0} \frac{\sin x - x}{x^3} = -\frac{1}{6}\,.$$

□

Problem 4. For given real numbers a, b, define function $f : \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \to \mathbb{R}$ by

$$f(x) = \begin{cases} (x - \sin x)\cos\dfrac{1}{x^2} & \text{if } -\dfrac{\pi}{2} < x < 0\,, \\ a\tan x + b & \text{if } 0 \leqslant x < \dfrac{\pi}{2}\,. \end{cases}$$

1. (3pts) Find all values of a and b such that f is continuous at 0.

2. (5pts) Find all values of a and b such that f is differentiable at 0.

3. (7pts) Can f be continuously differentiable on $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$?

Solution. 1. Using ($\star$) from Problem 3, we find that

$$\frac{x^3}{6} \leqslant \frac{x - \sin x}{x} \leqslant \frac{x^3}{6} - \frac{x^5}{120} \qquad \forall\, x < 0\,.$$

Therefore, by the fact that $\left|\cos\frac{1}{x}\right| \leqslant 1$ for all $x \neq 0$,

$$\left|f(x)\right| \leqslant \frac{x^3}{6} - \frac{x^5}{120} \qquad \forall\, -1 < x < 0\,.$$

The Squeeze Theorem then implies that $\lim\limits_{x\to 0^-} f(x) = 0$.

Suppose that f is continuous at 0. Then $\lim\limits_{x\to 0} f(x) = f(0)$; thus

$$0 = \lim_{x\to 0^-} f(x) = f(0) = b = \lim_{x\to 0^+} f(x)$$

which implies that $b = 0$ (but a can be arbitrary) if and only if f is continuous at 0.

2. Using ($\star$) from Problem 3, we find that

$$\frac{x^2}{6} - \frac{x^4}{120} \leqslant \frac{x - \sin x}{x} \leqslant \frac{x^2}{6} \qquad \forall\, x < 0\,. \qquad (\star\star)$$

Therefore, by the fact that $\left|\cos\frac{1}{x}\right| \leqslant 1$ for all $x \neq 0$,

$$\left|\frac{f(x)}{x}\right| \leqslant \frac{x^2}{6} \qquad \forall -1 < x < 0\,.$$

Suppose that f is differentiable at 0. Then f is continuous at 0 which implies that $f(0) = b = 0$. Moreover,

$$\lim_{h\to 0^-}\frac{f(h)-f(0)}{h} = \lim_{h\to 0^+}\frac{f(h)-f(0)}{h} = \lim_{h\to 0^+}\frac{a\tan h - a}{h} = a\frac{d}{dx}\Big|_{x=0}\tan x = a\sec^2 0 = a\,.$$

On the other hand, using $(\star\star)$ we find that

$$\lim_{h\to 0^-}\frac{f(h)-f(0)}{h} = \lim_{h\to 0^-}\frac{f(h)}{h} = 0\,;$$

thus $a = b = 0$ if and only if f is differentiable at 0.

3. Suppose that f is continuously differentiable. Then f is differentiable at 0 which, using 2, implies that $a = b = 0$. Therefore,

$$f'(x) = \begin{cases} 0 & \text{if } x \geqslant 0\,,\\ (1-\cos x)\cos\frac{1}{x^2} + \frac{2(x-\sin x)}{x^3}\cos\frac{1}{x^2} & \text{if } x < 0\,. \end{cases}$$

Since f' is continuous at 0, we must have

$$\lim_{x\to 0^-} f'(x) = \lim_{x\to 0} f'(x) = f'(0) = 0\,.$$

On the other hand, using $(\star)$ in Problem 3, we find that the limit

$$\lim_{x\to 0^-}\frac{2(x-\sin x)}{x^3}\cos\frac{1}{x^2}\ \text{D.N.E.}$$

Since $\lim\limits_{x\to 0^-}(1-\cos x)\cos\frac{1}{x^2} = 0$ (by the Squeeze Theorem), we find that

$$\lim_{x\to 0^-} f(x)\ \text{D.N.E.}$$

which implies that f cannot be continuously differentiable on $\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$ no matter what a and b are. □

Problem 5. Suppose that y is an implicit function defined by the relation $\sin(x+y) = y^2\cos x$.

1. (5pts) Find $\frac{dy}{dx}$.
2. (10pts) Find the concavity of the graph of the implicit function y near the point $(\pi, 0)$.

Solution. 1. By implicit differentiation,

$$\cos(x+y)\cdot\left(1+\frac{dy}{dx}\right) = 2y\frac{dy}{dx}\cos x - y^2\sin x\,;$$

thus

$$\frac{dy}{dx} = \frac{\cos(x+y)+y^2\sin x}{2y\cos x - \cos(x+y)}\,. \tag{$\star\star$}$$

2. Using $(\star\star)$,

$$\frac{d^2y}{dx^2} = \frac{\left[-\sin(x+y)\left(1+\frac{dy}{dx}\right) + 2y\frac{dy}{dx}\sin x + y^2\cos x\right]\cdot\left[2y\cos x - \cos(x+y)\right])}{\left[2y\cos x - \cos(x+y)\right]^2}$$
$$- \frac{\left[\cos(x+y) + y^2\sin x\right]\cdot\left[2\frac{dy}{dx}\cos x - 2y\sin x + \sin(x+y)\left(1+\frac{dy}{dx}\right)\right]}{\left[2y\cos x - \cos(x+y)\right]^2}.$$

Therefore, by the fact that $\left.\frac{dy}{dx}\right|_{(x,y)=(\pi,0)} = -1$, we find that

$$\left.\frac{d^2y}{dx^2}\right|_{(x,y)=(\pi,0)} = \frac{0-(-1)\left[2\cdot(-1)\cdot(-1)-0+0\right]}{(0-1)^2} = 2 > 0\,;$$

thus the fact that y is three times differentiable we find that near $(\pi,\,0)$ we have $\frac{d^2\,y}{dx^2} > 0$. Therefore, the graph of the implicit function y is concave upward near $(\pi, 0)$. □

Problem 6. (12pts) Find all the slant asymptotes of the graph of the function

$$f(x) = \frac{x + \sqrt{9x^2 - 12x}}{2}.$$

Solution. For $x \neq 0$, $\frac{f(x)}{x} = \frac{1}{2} + \frac{\sqrt{9x^2-12x}}{2x}$.

1. If $x > 0$,

$$\frac{f(x)}{x} = \frac{1}{2} + \frac{1}{2}\sqrt{\frac{9x^2-12x}{x^2}} = \frac{1}{2} + \frac{1}{2}\sqrt{9-\frac{12}{x}}\,;$$

thus $\lim\limits_{x\to\infty}\frac{f(x)}{x} = \frac{1}{2} + \frac{\sqrt{9}}{2} = 2$. Moreover,

$$f(x) - 2x = \frac{-3x + \sqrt{9x^2-12x}}{2} = \frac{-12x}{2(3x+\sqrt{9x^2-12x})} = \frac{-6}{3+\sqrt{9-\frac{12}{x}}}.$$

Therefore,

$$\lim_{x\to\infty}\left[f(x) - 2x\right] = \frac{-6}{3+\sqrt{9}} = -1$$

which shows that $y = 2x - 1$ is an slant asymptote.

2. If $x < 0$,

$$\frac{f(x)}{x} = \frac{1}{2} - \frac{1}{2}\sqrt{\frac{9x^2-12x}{x^2}} = \frac{1}{2} - \frac{1}{2}\sqrt{9-\frac{12}{x}}\,;$$

thus $\lim\limits_{x\to-\infty}\frac{f(x)}{x} = \frac{1}{2} - \frac{\sqrt{9}}{2} = -1$. Moreover,

$$f(x) + x = \frac{3x + \sqrt{9x^2-12x}}{2} = \frac{12x}{2(3x-\sqrt{9x^2-12x})} = \frac{6}{3+\sqrt{9-\frac{12}{x}}}.$$

Therefore,

$$\lim_{x\to\infty}\left[f(x) - 2x\right] = \frac{6}{3+\sqrt{9}} = 1$$

which shows that $y = -x + 1$ is an slant asymptote. $\square$

Problem 7. Let $f : [-2\pi, 2\pi] \to \mathbb{R}$ be given by

$$f(x) = (18x - x^3)\sin x + (24 - 6x^2)\cos x\,.$$

1. (6pts) Find the inflection points of the graph of f.

2. (6pts) Use the first derivative test to find all the relative extrema of f'.

3. (6pts) Show that $|f(x) - f(y)| \leqslant (8\pi^3 - 12\pi)|x - y|$ for all $x, y \in [-2\pi, 2\pi]$.

Solution. First we compute f' and f'' as follows: by the product rule,

$$\begin{aligned} f'(x) &= (18 - 3x^2)\sin x + (18x - x^3)\cos x - 12x\cos x - (24 - 6x^2)\sin x \\ &= (3x^2 - 6)\sin x + (6x - x^3)\cos x\,, \end{aligned}$$

thus

$$f''(x) = 6x\sin x + (3x^2 - 6)\cos x + (6 - 3x^2)\cos x - (6x - x^3)\sin x = x^3\sin x\,.$$

1. By the fact that $f''(x) = x^3\sin x$, we find that $f''(x) = 0$ if and only if $x = 0$ or $x = \pm\pi$. Since f'' changes from negative to positive at $-\pi$, changes from positive to negative at π, and remains positive on both sides of 0, we find that the inflection points of the graph of f are

$$(\pi, 6\pi^2 - 24), (-\pi, 6\pi^2 - 24)\,.$$

2. Since f'' changes from negative to positive at $-\pi$, changes from positive to negative at π, and remains positive on both sides of 0, we find that f' attains its relative minimum at $-\pi$ and attains its relative maximum at π. Therefore, $f'(-\pi) = 6\pi - \pi^3$ is a relative minimum of f' on $[-2\pi, 2\pi]$ and $f'(\pi) = \pi^3 - 6\pi$ is a relative maximum of f' on $[-2\pi, 2\pi]$.

3. Since

$$\big|f'(2\pi)\big| = \big|f'(-2\pi)\big| = 8\pi^3 - 12\pi \qquad \text{and} \qquad \big|f'(\pi)\big| = \big|f'(-\pi)\big| = \pi^3 - 6\pi$$

and $8\pi^3 - 12\pi > \pi^3 - 6\pi$, we find that

$$\max_{x\in[-2\pi,2\pi]} \big|f'(x)\big| \leqslant 8\pi^3 - 12\pi\,;$$

thus the mean value theorem implies that

$$\big|f(x) - f(y)\big| \leqslant (8\pi^3 - 12\pi)|x - y| \qquad \forall\, x, y \in [-2\pi, 2\pi]\,.$$

$\square$

Problem 8. (15pts) 某同學被要求寫下框中敘述

> Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable function satisfying that
> $$f'(x) = f(x) \qquad \text{for all } x \in \mathbb{R}. \tag{$\star$}$$
> If $f(0) = 1$, then f is increasing on $\mathbb{R}$.

的證明。該同學證明未完成但如下，請於分別敘述 (A) 定理與 (B) 定理的名稱與定理內容，並幫助該同學完成此題證明之 (C) 與 (D) 部份。

Proof. 因為 f 在實數軸上可微，所以 f 在實數軸上連續；再因為 $f(0) > 0$，由 f 的連續性必存在一個 $\delta > 0$ 使得只要 x 落在 $(-\delta, \delta)$ 區間我們一定有 $f(x) > 0$。由 $(\star)$ 這個條件我們得知只要 x 落在 $(-\delta, \delta)$ 區間，必有 $f'(x) > 0$。因此，f 在 $(-\delta, \delta)$ 區間上嚴格遞增。

接下來我們使用矛盾證法分開證明 f 在 $[0, \infty)$ 與在 $(-\infty, 0]$ 上遞增。

1. 假設 f 在 $[0, \infty)$ 不遞增，則必有一正實數 c 使得 $f(c) = f'(c) < 0$。因為 f 在 $[0, c]$ 閉區間上連續，由 __(A)__ 定理可知 f 在 $[0, c]$ 閉區間上取得到最大值。假設在 $[0, c]$ 區間中的某點 x_0 函數 f 取到在 $[0, c]$ 區間上的最大值（亦即 $f(x_0)$ 為 f 在 $[0, c]$ 區間上的最大值）。
   (a) 如果 $0 < x_0 < c$，則由 __(B)__ 定理我們得知 $f'(x_0) = 0$。再由 $(\star)$ 這個條件我們得知 $f(x_0) = 0$；然而 $f(0) = 1 > 0 = f(x_0)$，$f(x_0)$ 不可能為 f 在 $[0, c]$ 區間中的最大值。
   (b) 因為 f 在 $(-\delta, \delta)$ 區間中嚴格遞增，我們得知 $f(0)$ 不可能為 f 在 $[0, c]$ 中的最大值。所以剩下一個可能性是 $x_0 = c$。然而 $f(c) < 0$ 且 $f(0) = 1 > 0 > f(c)$，我們得知 $f(c)$ 也不可能為 f 在 $[0, c]$ 中的最大值。

   由上述兩點得知 f 這個連續函數不可能在 $[0, c]$ 上取得最大值這個矛盾。因此，f 在 $[0, \infty)$ 區間上遞增。

2. 假設 f 在 $(-\infty, 0]$ 不遞增，則必有一負實數 d 使得 $f(d) = f'(d) < 0$。由 f 的連續性，存在一個 $\delta > 0$ 使得 f 在 $[d, d+\delta]$ 區間上小於零，再由 $(\star)$ 這個條件我們得知 f' 在 $[d, d+\delta]$ 區間上小於零，所以 f 在 $[d, d+\delta]$ 區間上嚴格遞減。接下來，由於 f 在 $[d, 0]$ 上連續，由 __(A)__ 定理知 f 在 $[d, 0]$ 閉區間上取得到最小值。假設在 $[d, 0]$ 區間中的某點 x_1 函數 f 取到在 $[d, 0]$ 區間上的最小值（亦即 $f(x_1)$ 為 f 在 $[d, 0]$ 區間上的最小值）。試模仿 1(a)(b) 得到 f 這個連續函數不可能在 $[d, 0]$ 上取得最小值這個矛盾。

   (C)

最後證明如果 f 在 $[0, \infty)$ 上遞增且 $f : (-\infty, 0]$ 上遞增，則 f 在 $\mathbb{R}$ 上遞增。

(D)

□