Exercise Problem Sets 14

Dec. 27. 2019

Problem 1. Find at least two ways to compute the following integrals.

1. $\displaystyle\int \frac{x-1}{x^2-4x-5}\,dx$ 2. $\displaystyle\int \frac{3x^2-2}{x^3-2x-1}\,dx$ 3. $\displaystyle\int \frac{1+4\cot x}{4-\cot x}\,dx$

4. $\displaystyle\int \frac{1}{x(x^4+1)}\,dx$ 5. $\displaystyle\int \frac{4}{\tan x-\sec x}\,dx$

Problem 2. Find the following indefinite integrals using the techniques of partial fractions.

1. $\displaystyle\int \frac{x}{x^4-1}\,dx$ 2. $\displaystyle\int \frac{x}{x^4+4x^2+3}\,dx$ 3. $\displaystyle\int \frac{x^3+1}{x^3-x^2}\,dx$ 4. $\displaystyle\int \frac{x-1}{x^2-4x+5}\,dx$

5. $\displaystyle\int \frac{1}{x^6+1}\,dx$ 6. $\displaystyle\int \frac{1}{(x-2)(x^2+4)}\,dx$ 7. $\displaystyle\int \frac{1}{x+4+4\sqrt{x+1}}\,dx$ 8. $\displaystyle\int \frac{1}{x\sqrt{4x+1}}\,dx$

9. $\displaystyle\int \frac{1}{x^2\sqrt{4x+1}}\,dx$ 10. $\displaystyle\int \frac{1}{x+\sqrt[3]{x}}\,dx$ 11. $\displaystyle\int \frac{1}{1+2e^x-e^{-x}}\,dx$ 12. $\displaystyle\int \frac{1}{e^{3x}-e^x}\,dx$

13. $\displaystyle\int \frac{\sin x\cos x}{\sin^4 x+\cos^4 x}\,dx$ 14. $\displaystyle\int \frac{1}{3-2\sin x}\,dx$ 15. $\displaystyle\int \frac{1}{1+\sin\theta+\cos\theta}\,d\theta$ 16. $\displaystyle\int \sqrt{\tan x}\,dx$

Problem 3. Determine if the following improper integral converges or not.

1. $\displaystyle\int_0^\infty \frac{dx}{\sqrt[3]{x^4-x^2}}$. 2. $\displaystyle\int_1^\infty \frac{dx}{x(\ln x)^\alpha}$ for $\alpha>0$. 3. $\displaystyle\int_1^\infty \frac{\ln x}{x^\alpha}\,dx$. 4. $\displaystyle\int_{100}^\infty \frac{dx}{x(\ln\ln x)^\alpha}$ for $\alpha>0$.

5. $\displaystyle\int_0^\pi \frac{dx}{\sqrt{x}+\sin x}$. 6. $\displaystyle\int_0^\pi \frac{dx}{x-\sin x}$. 7. $\displaystyle\int_0^{\ln 2} x^{-2}e^{-\frac{1}{x}}\,dx$. 8. $\displaystyle\int_0^1 \frac{e^{-\sqrt{x}}}{\sqrt{x}}\,dx$. 9. $\displaystyle\int_1^\infty \frac{dx}{\sqrt{e^x-x}}$.

10. $\displaystyle\int_{-\infty}^\infty \frac{dx}{e^x+e^{-x}}$. 11. $\displaystyle\int_\pi^\infty \frac{1+\sin x}{x^2}\,dx$. 12. $\displaystyle\int_{-1}^1 \ln|x|\,dx$.

Problem 4. Show that if $a>-1$ and $b>a+1$, then the integral $\displaystyle\int_0^\infty \frac{x^a}{1+x^b}\,dx$ is convergent.

Problem 5. If $f:[0,\infty)\to\mathbb{R}$ is continuous, the Laplace transform of f, denoted by $\mathscr{L}(f)$, is the function defined by

$$\mathscr{L}(f)(s)=\int_0^\infty e^{-st}f(t)\,dt\,,$$

and the domain of $\mathscr{L}(f)$ is the set consisting of all numbers s for which the integral converges.

1. Find the Laplace transforms of the following functions.

   (i) $f(t)=t^k$, where $k\in\mathbb{N}\cup\{0\}$ (ii) $f(t)=e^{at}\sin(bt)$ (iii) $f(t)=e^{at}\cos(bt)$.

2. Let $k\in\mathbb{N}$ and $f:[0,\infty)\to\mathbb{R}$ is k-times continuously differentiable. Show that if $\mathscr{L}(f)$ exists for $s>a$, then

$$\mathscr{L}(f^{(k)})(s)=s^k\mathscr{L}(f)(s)-s^{k-1}f(0)-s^{k-2}f'(0)-\cdots-sf^{(k-2)}(0)-f^{(k-1)}(0)\quad\text{for } s>a\,.$$

3. Suppose that m, k are positive numbers, b, ω are non-negative numbers, and $y : [0, \infty) \to \mathbb{R}$ is a twice differentiable function satisfying

$$my''(t) + by'(t) + ky(t) = \sin(\omega t) \qquad y(0) = y_0 \text{ and } y'(0) = y_1 .$$

Find $\mathscr{L}(y)$.

4. Suppose that you know that $\mathscr{L}$ is one-to-one, find y in the previous problem with the parameters $(m, b, k, y_0, y_1) = (1, 2, 5, 1, 0)$.

Problem 6. In this problem we intend to compute

$$\int_0^{\frac{\pi}{2}} \ln \sin x \, dx .$$

1. Determine whether the integral above is an improper integral or not.

2. Prove the identities

$$\int_0^{\pi} \ln \sin x \, dx = 2 \int_0^{\frac{\pi}{2}} \ln \sin(2x) \, dx$$

and

$$\int_0^{\frac{\pi}{2}} \ln \sin x \, dx = \int_0^{\frac{\pi}{2}} \ln \cos x \, dx .$$

3. Find $\displaystyle\int_0^{\frac{\pi}{2}} \ln \sin x \, dx$ (using identities in 2).

Problem 7. In the following, we are going to compute definite (improper) integrals using the techniques of introducing a new variable t in a suitable. For example, to compute the integral

$$\int_0^1 \frac{x-1}{\ln x} \, dx ,$$

we define $I(t) = \displaystyle\int_0^1 \frac{x^t - 1}{\ln x} \, dx$. Assume that

$$\lim_{h \to 0} \int_0^1 \frac{x^{t+h} - x^t}{h \ln x} \, dx = \int_0^1 \lim_{h \to 0} \frac{x^{t+h} - x^t}{h \ln x} \, dx .$$

Then by the fact that

$$\lim_{h \to 0} \frac{x^{t+h} - x^t}{h} = x^t \ln x ,$$

we find that

$$I'(t) = \lim_{h \to 0} \frac{I(t+h) - I(t)}{h} = \lim_{h \to 0} \int_0^1 \frac{x^{t+h} - x^t}{h \ln x} \, dx \overset{\text{(assumption)}}{=} \int_0^1 \lim_{h \to 0} \frac{x^{t+h} - x^t}{h \ln x} \, dx = \int_0^1 x^t \, dx .$$

Since we are going to compute $I(1)$, we may assume that $t > -1$ and find that under our assumption,

$$I'(t) = \int_0^1 x^t \, dx = \frac{x^{t+1}}{t+1}\Big|_{x=0}^{x=1} = \frac{1}{t+1} ;$$

thus $I(t) = \ln(t+1) + C$. Since $I(0) = 0$, we find that $C = 0$ which further implies that $I(1) = \ln 2$.

In the following, we are going to assume that the change of order of taking limit (such as $\lim\limits_{h \to 0}$) and integration $\left(\text{such as } \displaystyle\int_0^1\right)$ does not affect the outcome of the computations.

1. Compute $\displaystyle\int_0^1 \frac{\ln(x+1)}{x^2+1}\,dx$.

   Hint: Let $I(t) = \displaystyle\int_0^1 \frac{\ln(tx+1)}{x^2+1}\,dx$. Compute $I'(t)$ by the assumption above and find the integral using the techniques of partial fractions.

2. Compute $\displaystyle\int_0^\infty \frac{\sin x}{x}\,dx$.

   Hint: Let $I(t) = \displaystyle\int_0^\infty \frac{e^{-tx}\sin x}{x}\,dx$. Compute $I'(t)$ by the assumption above and use the fact that $\displaystyle\lim_{t\to\infty} I(t) = 0$.