Exercise Problem Sets 13

Dec. 20. 2019

Problem 1. Find the following indefinite integrals.

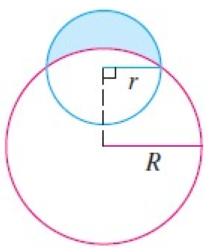
1. $\displaystyle\int x\csc x\cot x\,dx$
2. $\displaystyle\int \frac{\sqrt{1+\ln x}}{x\ln x}\,dx$
3. $\displaystyle\int x\sin^2 x\,dx$
4. $\displaystyle\int \exp(\sqrt[3]{x})\,dx$
5. $\displaystyle\int x\arcsin x\,dx$
6. $\displaystyle\int x\arctan x\,dx$
7. $\displaystyle\int x^2\arctan x\,dx$
8. $\displaystyle\int \ln(x^2-1)\,dx$
9. $\displaystyle\int \sin\sqrt{ax}\,dx$
10. $\displaystyle\int x\tan^2 x\,dx$
11. $\displaystyle\int x^5e^{-x^3}\,dx$
12. $\displaystyle\int \frac{x\ln x}{\sqrt{x^2-1}}\,dx$
13. $\displaystyle\int \sqrt{x}e^{\sqrt{x}}\,dx$
14. $\displaystyle\int \frac{\arctan\sqrt{x}}{\sqrt{x}}\,dx$
15. $\displaystyle\int \frac{\ln(x+1)}{x^2}\,dx$
16. $\displaystyle\int x\sin^2 x\cos x\,dx$
17. $\displaystyle\int \frac{dx}{x^4\sqrt{x^2-2}}$

Problem 2. The function $y=e^{x^2}$ and $y=x^2e^{x^2}$ don't have elementary anti-derivatives, but $y=(2x^2+1)e^{x^2}$ does. Find the indefinite integral $\displaystyle\int(2x^2+1)e^{x^2}\,dx$.

Problem 3. Obtain a recursive formula for $\displaystyle\int x^p(ax^n+b)^q\,dx$ and use this relation to find the indefinite integral $\displaystyle\int x^3(x^7+1)^4\,dx$.

Problem 4. Obtain a recursive formula for $\displaystyle\int x^m(\ln x)^n\,dx$ and use this relation to find the indefinite integral $\displaystyle\int x^4(\ln x)^3\,dx$.

Problem 5. Find the area of the crescent-shaped region (called a lune) bounded by arcs of circles with radii r and R. (See the figure)



Problem 6. Complete the following.

1. Let $f:[a,b]\to[c,d]$ be a continuously differentiable increasing function. Suppose that f has an inverse f^{-1}. Show that

$$\int_a^b f(x)\,dx+\int_c^d f^{-1}(y)\,dy=bf(b)-af(a)\,. \qquad (\star)$$

2. How about if f is decreasing?

3. Use ($\star$) to compute $\int_0^1 \arcsin x \, dx$ and $\int_0^1 \arctan x \, dx$.

4. Let F be an anti-derivative of a continuously differentiable function f with inverse f^{-1}. Find an anti-derivative of f^{-1} in terms of f and F.

Problem 7. For $n \in \mathbb{N} \cup \{0\}$, the Legendre polynomial of degree n, denoted by P_n, is defined by

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2 - 1)^n .$$

1. Show that $\int_{-1}^1 P_n(x) P_m(x) \, dx = 0$ if $m \neq n$.

2. Show that $\int_{-1}^1 P_n(x)^2 \, dx = \frac{2}{2n+1}$ for all $n \in \mathbb{N} \cup \{0\}$.

3. Show that $\int_{-1}^1 x^m P_n(x) \, dx = 0$ if $m < n$.

4. Evaluate $\int_{-1}^1 x^n P_n(x) \, dx$.

Problem 8. Let $\alpha_1, \alpha_2, \cdots, \alpha_n$ be distinct real numbers, and

$$g(x) = \prod_{k=1}^n (x - \alpha_k) \equiv (x - \alpha_1)(x - \alpha_2) \cdots (x - \alpha_n) .$$

Use the partial fraction expansion to prove Newton's formula

$$\frac{\alpha_1^k}{g'(\alpha_1)} + \frac{\alpha_2^k}{g'(\alpha_2)} + \cdots + \frac{\alpha_n^k}{g'(\alpha_n)} = \begin{cases} 0 & \text{for } k = 0, 1, 2, \cdots, n-2 , \\ 1 & \text{for } k = n-1 , \end{cases}$$

Hint: By partial fraction, for $k < n - 1$

$$\frac{x^k}{(x - \alpha_2)(x - \alpha_3) \cdots (x - \alpha_n)} = \frac{A_2}{x - \alpha_2} + \frac{A_3}{x - \alpha_3} + \cdots + \frac{A_n}{x - \alpha_n} .$$

Show that $A_j = \frac{\alpha_j^k (\alpha_j - \alpha_1)}{g'(\alpha_j)}$ and conclude from here. Do the same for the case $k = n - 1$.