Exercise Problem Sets 10

Nov. 22. 2019

Problem 1. Let I be an interval, and $f : I \to \mathbb{R}$ be one-to-one, onto and continuous. Show that if $g : \mathbb{N} \to \mathbb{R}$ is a function satisfying that $\lim_{n\to\infty} f(g(n)) = b$, then $\lim_{n\to\infty} g(n) = f^{-1}(b)$.

Problem 2. Show that the following functions (defined by integrals) are one-to-one and find $(f^{-1})'(0)$.

1. $f(x) = \displaystyle\int_2^x \sqrt{1+t^2}\,dt.$ 2. $f(x) = \displaystyle\int_2^x \frac{dt}{\sqrt{1+t^4}}.$

Problem 3. Let f be an one-to-one, twice differentiable function with an inverse function g.

1. Show that g is twice differentiable function and find g''.
2. Show that if in addition f is strictly increasing and the graph of f is concave upward, then the graph of g is concave downward.

Problem 4. Find the limit $\lim_{n\to\infty} \left(\frac{n!}{n^n}\right)^{\frac{1}{n}}$ through the following steps.

1. Show that $\displaystyle\sum_{k=1}^{n-1} \frac{1}{n} \ln\frac{k}{n} \leqslant \int_{\frac{1}{n}}^{1} \ln x\,dx \leqslant \sum_{k=2}^{n} \frac{1}{n} \ln\frac{k}{n}.$
2. Find $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{n} \ln\frac{k}{n}$.
3. Find $\lim_{n\to\infty} \left(\frac{n!}{n^n}\right)^{\frac{1}{n}}$.

Hint: 1. Use the property of integrals.
3. Using problem 1.

Problem 5. Show that for all natural number n,

$$\sum_{k=1}^{2n} \frac{(-1)^{k-1}x^k}{k} \leqslant \ln(1+x) \leqslant \sum_{k=1}^{2n-1} \frac{(-1)^{k-1}x^k}{k} \qquad \forall\, x > 0\,.$$

Problem 6. Find the derivative of the following functions by first taking the logarithm (base e) and then differentiating.

1. $y = \dfrac{x(x-1)^{\frac{3}{2}}}{\sqrt{x+1}}$, $x > 1$. 2. $y = \dfrac{(x+1)(x-2)}{(x-1)(x+2)}$, $x > 2$

Problem 7. Use implicit differentiation to find $\dfrac{dy}{dx}$, where (x, y) satisfies the relation $4xy + \ln x^2 y = 7$.

Problem 8. Locate any relative extrema and points of inflection of the function $y = x^2 \ln\dfrac{x}{4}$.

Problem 9. Use the substitution of variable $t = \tan\dfrac{x}{2}$ to find the integral $\displaystyle\int \csc x\,dx$.

Problem 10. Find the following indefinite integrals.

1. $\displaystyle\int \frac{(\ln x)^2}{x}\,dx.$
2. $\displaystyle\int \frac{\ln\sqrt{x}}{x}\,dx.$
3. $\displaystyle\int \frac{dx}{x(\ln x^2)^3}.$
4. $\displaystyle\int \frac{(1+\ln x)^2}{x}\,dx.$
5. $\displaystyle\int \frac{\sin(\ln x)}{x}\,dx.$
6. $\displaystyle\int \frac{\sin 2x}{1+\cos^2 x}\,dx.$

Problem 11. Show that $\dfrac{1}{y} < \dfrac{\ln x - \ln y}{x-y} < \dfrac{1}{x}$ for all $0 < x < y$.