微積分 MA1002-A 上課筆記（精簡版）

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For a double integral $\iint\limits_R f(x,y)\,dA$ which can be computed by the iterated integral $\int\Big(\int f(x,y)dy\Big)\,dx$, with a one-to-one change of variables $x = x(u,v)$ and $y = y(u,v)$, we obtain that

$$\iint\limits_R f(x,y)\,dA = \iint\limits_{R'} f(x(u,v),y(u,v))\left|\begin{vmatrix} x_u(u,v) & x_v(u,v) \\ y_v(u,v) & y_u(u,v)\end{vmatrix}\right| dA',$$

where R' is a region $(x,y)(R') = R$.

For a triple integral $\iiint\limits_Q f(x,y,z)\,dV$ which can be computed by the iterated integral $\int\Big[\int\Big(\int f(x,y,z)dz\Big)dy\Big]dx$, with a one-to-one change of variables $x = x(u,v,w)$, $y = y(u,v,w)$ and $z = z(u,v,w)$, we obtain that

$$\begin{aligned}&\iiint\limits_Q f(x,y,z)\,dV \\ &\quad = \iiint\limits_{Q'} f(x(u,v,w),y(u,v,w),z(u,v,w))\left|\begin{vmatrix} x_u(u,v,w) & x_v(u,v,w) & x_w(u,v,w) \\ y_u(u,v,w) & y_v(u,v,w) & y_w(u,v,w) \\ z_u(u,v,w) & z_v(u,v,w) & z_w(u,v,w)\end{vmatrix}\right| dV'\end{aligned}$$

The naive (but **wrong**) computations above motivate the following

Definition 14.20

If $x = x(u,v)$ and $y = y(u,v)$, the ***Jacobian*** of x and y with respect to u and v, denoted by $\dfrac{\partial(x,y)}{\partial(u,v)}$, is

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = x_u y_v - x_v y_u\,.$$

If $x = x(u,v,w)$, $y = y(u,v,w)$ and $z = z(u,v,w)$, the ***Jacobian*** of x, y and z with respect to u, v and w, denoted by $\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$, is

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} x_u & x_v & x_w \\ y_u & y_v & y_w \\ z_u & z_v & z_w \end{vmatrix} = x_u y_v z_w + x_w y_u z_v + x_v y_w z_u - x_w y_v z_u - x_v y_u z_w - x_u y_w z_v\,.$$

In general, if $g_1, g_2, \cdots, g_n$ are functions of n-variables (whose variables are denoted by $u_1, u_2, \cdots, u_n$), then the Jacobian of $g_1, g_2, \cdots, g_n$ (with respect to $u_1, u_2, \cdots, u_n$), denoted by $\dfrac{\partial(g_1,\cdots,g_n)}{\partial(u_1,\cdots,u_n)}$, is

$$
\frac{\partial(g_1,\cdots,g_n)}{\partial(u_1,\cdots,u_n)} = \begin{vmatrix} \dfrac{\partial g_1}{\partial u_1} & \dfrac{\partial g_1}{\partial u_2} & \cdots & \dfrac{\partial g_1}{\partial u_n} \\ \dfrac{\partial g_2}{\partial u_1} & \dfrac{\partial g_2}{\partial u_2} & \cdots & \dfrac{\partial g_2}{\partial u_n} \\ \vdots & \vdots & \ddots & \vdots \\ \dfrac{\partial g_n}{\partial u_1} & \dfrac{\partial g_n}{\partial u_2} & \cdots & \dfrac{\partial g_n}{\partial u_n} \end{vmatrix}.
$$

Example 14.21. The Jacobian of the change of variables given by the spherical coordinate $x = \rho\cos\theta\sin\phi$, $y = \rho\sin\theta\sin\phi$, $z = \rho\cos\phi$ is

$$
\begin{aligned}
\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} &= \begin{vmatrix} \cos\theta\sin\phi & -\rho\sin\theta\sin\phi & \rho\cos\theta\cos\phi \\ \sin\theta\sin\phi & \rho\cos\theta\sin\phi & \rho\sin\theta\cos\phi \\ \cos\phi & 0 & -\rho\sin\phi \end{vmatrix} \\
&= -\rho^2\cos^2\theta\sin^3\phi - \rho^2\sin^2\theta\sin\phi\cos^2\phi - \rho^2\cos^2\theta\sin\phi\cos^2\phi - \rho^2\sin^2\theta\sin^3\phi \\
&= -\rho^2\cos^2\theta\sin\phi - \rho^2\sin^2\theta\sin\phi = -\rho^2\sin\phi\,.
\end{aligned}
$$

The Jacobian of the change of variables given by the cylindrical coordinate $x = r\cos\theta$, $y = r\sin\theta$, $z = z$ is

$$
\frac{\partial(x,y,z)}{\partial(r,\theta,z)} = \begin{vmatrix} \cos\theta & -r\sin\theta & 0 \\ \sin\theta & r\cos\theta & 0 \\ 0 & 0 & 1 \end{vmatrix} = r\,.
$$

Even though the derivation of the change of variables is wrong; however, the conclusion is in fact correct, and we have the following

Theorem 14.22

Let $\mathrm{O} \subseteq \mathbb{R}^2$ be an open set that has area, and $g = (g_1, g_2) : \mathrm{O} \to \mathbb{R}^2$ be an one-to-one continuously differentiable function such that g^{-1} is also continuously differentiable. Assume that the Jacobian of g_1, g_2 (with respective to their variables) does not vanish in O. If $f : g(\mathrm{O}) \to \mathbb{R}$ is integrable (on $g(\mathrm{O})$), then

$$
\iint\limits_{g(\mathrm{O})} f(x,y)\,dA = \iint\limits_{\mathrm{O}} f\big(g_1(u,v), g_2(u,v)\big)\left|\frac{\partial(g_1,g_2)}{\partial(u,v)}\right| dA'\,,
$$

where the integral on the right-hand side is the double integral of the function $f\big(g_1(u,v), g_2(u,v)\big)\left|\dfrac{\partial(g_1,g_2)}{\partial(u,v)}\right|$ (with variables u, v) on O.

Theorem 14.23

Let $\mathrm{O} \subseteq \mathbb{R}^3$ be an open set that has volume (that is, the constant function is Riemann integrable on O), and $g = (g_1, g_2, g_3) : \mathrm{O} \to \mathbb{R}^3$ be an one-to-one continuously differentiable function such that g^{-1} is also continuously differentiable. Assume that the Jacobian of g_1, g_2, g_3 (with respective to their variables) does not vanish in O. If $f : g(\mathrm{O}) \to \mathbb{R}$ is integrable (on $g(\mathrm{O})$), then

$$\iiint\limits_{g(\mathrm{O})} f(x, y, z)\, dV = \iiint\limits_{\mathrm{O}} f\big(g_1(u, v, w), g_2(u, v, w), g_3(u, v, w)\big) \Big|\frac{\partial(g_1, g_2, g_3)}{\partial(u, v, w)}\Big|\, dV',$$

where the integral on the right-hand side is the triple integral of the function $f\big(g_1(u, v, w), g_2(u, v, w), g_3(u, v, w)\big)\Big|\frac{\partial(g_1, g_2, g_3)}{\partial(u, v, w)}\Big|$ (with variables u, v, w) on O.

Remark 14.24. Suppose that O is an open set in the plane such that the boundary of O, denoted by $\partial\mathrm{O}$, has zero area. Under suitable assumptions (for example, if the set of discontinuities of f has zero area and f is bounded above or below by a constant), we have

$$\iint\limits_{\mathrm{O}} f(x, y)\, dA = \iint\limits_{\bar{\mathrm{O}}} f(x, y)\, dA\,. \tag{14.5.1}$$

Example 14.25. Let $B = \big\{(x, y) \,\big|\, x^2 + y^2 < R^2\big\} - [0, 1) \times \{0\}$. Then the polar coordinate $x = x(r, \theta) = r\cos\theta$ and $y = y(r, \theta) = r\cos\theta$ is an one-to-one continuously differentiable function from $\mathrm{O} \equiv (0, R) \times (0, 2\pi) \to \mathbb{R}^2$ and the inverse function $r = r(x, y) = \sqrt{x^2 + y^2}$ and

$$\theta = \theta(x, y) = \begin{cases} \arccos\dfrac{x}{\sqrt{x^2 + y^2}} & \text{if } y > 0\,, \\ \pi & \text{if } y = 0\,, \\ 2\pi - \arccos\dfrac{x}{\sqrt{x^2 + y^2}} & \text{if } y < 0\,, \end{cases}$$

is also continuously differentiable (which you proved in Quiz). Therefore, the change of variables formula implies that

$$\iint\limits_{B} f(x, y) dA = \iint\limits_{(0,R)\times(0,2\pi)} f(r\cos\theta, r\sin\theta) r\, dA\,.$$

Let $D(R) = \big\{(x, y) \,\big|\, x^2 + y^2 \leqslant R^2\big\}$. Then $D = B \cup \partial B$ and $[0, R] \times [0, 2\pi] = (0, R) \times$

$(0,2\pi) \cup \partial\big[(0,R)\times(0,2\pi)\big]$; thus (14.5.1) further implies that

$$\iint\limits_{D(R)} f(x,y)\,dA = \iint\limits_{[0,R]\times[0,2\pi]} f(r\cos\theta, r\sin\theta)r\,dA\,.$$

In general, if a region R, in polar coordinate, can be expressed as

$$R' = \big\{(r,\theta)\,\big|\, a \leqslant \theta \leqslant b, g_1(\theta) \leqslant r \leqslant g_2(\theta)\big\}\,,$$

then

$$\iint\limits_{R} f(x,y)\,dA = \int_a^b \Big(\int_{g_1(\theta)}^{g_2(\theta)} f(r\cos\theta, r\sin\theta)r\,dr\Big)\,d\theta\,,$$

while if R, in polar coordinate, can be expressed as

$$R' = \big\{(r,\theta)\,\big|\, c \leqslant r \leqslant d, h_1(r) \leqslant \theta \leqslant h_2(r)\big\}\,,$$

then

$$\iint\limits_{R} f(x,y)\,dA = \int_c^d \Big(\int_{h_1(r)}^{h_2(r)} f(r\cos\theta, r\sin\theta)r\,d\theta\Big)\,dr\,.$$

Example 14.26. In this example we compute the double integral $\displaystyle\iint\limits_{R} \sqrt{1+4x^2+4y^2}\,dA$ that appears in Example 14.13, where $R = \big\{(x,y)\,\big|\, x^2+y^2 \leqslant 1\big\}$.

Using the polar coordinate, R can be expressed as $\big\{(r,\theta)\,\big|\, 0 \leqslant r \leqslant 1, 0 \leqslant \theta \leqslant 2\pi\big\}$; thus

$$\iint\limits_{R} \sqrt{1+4x^2+4y^2}\,dA = \int_0^{2\pi}\Big(\int_0^1 \sqrt{1+4r^2}\cdot r\,dr\Big)d\theta = \int_0^{2\pi}\Big[\frac{1}{12}(1+4r^2)^{\frac{3}{2}}\Big]\Big|_{r=0}^{r=1}\,d\theta$$

$$= \frac{1}{12}\int_0^{2\pi}(5\sqrt{5}-1)\,d\theta = \frac{\pi}{6}(5\sqrt{5}-1)\,.$$

Example 14.27. Let S be the subset of the upper hemisphere $z = \sqrt{1-x^2-y^2}$ enclosed by the curve C shown in the figure below

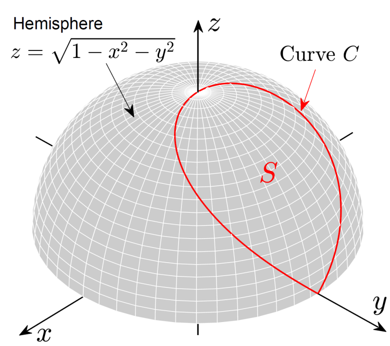


Figure 14.5: Curve S on the upper hemisphere

where each point of C corresponds to some point $(\cos t\sin t, \sin^2 t, \cos t)$ with $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Find the surface of S.

Let (x, y) be a boundary point of R. The $(x, y) = (\cos t\sin t, \sin^2 t)$ for some $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$; thus

$$x^2 + y^2 = \cos^2 t\sin^2 t + \sin^4 t = (\cos^2 t + \sin^2 t)\sin^2 t = \sin^2 t = y\,.$$

Therefore, the boundary of R consists of points (x, y) satisfying $x^2 + y^2 = y$ which shows that R is a disk centered at $\left(0, \frac{1}{2}\right)$ with radius $\frac{1}{2}$. Therefore,

$$R = \left\{(x, y) \,\middle|\, 0 \leqslant y \leqslant 1, -\sqrt{y - y^2} \leqslant x \leqslant \sqrt{y - y^2}\right\},$$

and by Theorem **??** the surface area of S is given by $\displaystyle\iint\limits_R \frac{1}{\sqrt{1 - x^2 - y^2}}\,dA$.

Now we apply the change of variables using the polar coordinates to compute this double integral. Since we have found the Jacobian of this change of variables, we only need to find the corresponding region R' of R in the $r\theta$-plane and the change of variables formula shows that the surface area of S is $\displaystyle\iint\limits_{R'} \frac{r}{\sqrt{1 - r^2}}\,dA'$.

By the fact that the boundary of R' maps to the boundary of R under the change of variables $x = r\cos\theta$ and $y = r\sin\theta$, we find that if (r, θ) is a boundary point of R', then (r, θ) satisfies $r^2 = r\sin\theta$; thus the boundary of R' consists of points (r, θ) satisfying $r = \sin\theta$ or $r = 0$ in the $r\theta$-plane. Since R locates on the upper half plane, $0 \leqslant \theta \leqslant \pi$, and the center of the disk R corresponds to point $\left(\frac{1}{2}, \frac{\pi}{2}\right)$ in the $r\theta$-plane, we conclude that

$$R' = \left\{(r, \theta) \,\middle|\, 0 \leqslant \theta \leqslant \pi, 0 \leqslant r \leqslant \sin\theta\right\}.$$

Therefore,

$$\begin{aligned}\iint\limits_{R'} \frac{r}{\sqrt{1 - r^2}}\,dA' &= \int_0^\pi \Big(\int_0^{\sin\theta} \frac{1}{\sqrt{1 - r^2}}\,r dr\Big) d\theta = \int_0^\pi \Big[\big(-\sqrt{1 - r^2}\big)\Big|_{r=0}^{r=\sin\theta}\Big]\,d\theta \\ &= \int_0^\pi \big(1 - |\cos\theta|\big)\,d\theta = \pi - 2\int_0^{\frac{\pi}{2}} \cos\theta\,d\theta = \pi - 2\Big(\sin\theta\Big|_{\theta=0}^{\theta=\frac{\pi}{2}}\Big) = \pi - 2\,.\end{aligned}$$

Example 14.28. In this example we compute the improper integral $\displaystyle\int_0^\infty e^{-x^2}\,dx$. First we note that this improper integral converges since $0 \leqslant e^{-x^2} \leqslant e^{-x}$ for all $x \geqslant 1$ and $\displaystyle\int_1^\infty e^{-x}\,dx = e^{-1} < \infty$, the comparison test implies that $\displaystyle\int_1^\infty e^{-x^2}\,dx$ converges.

Let $I = \displaystyle\int_0^\infty e^{-x^2}\,dx$. Then $I = \displaystyle\int_0^\infty e^{-y^2}\,dy$; thus

$$\begin{aligned} I^2 &= \Big(\int_0^\infty e^{-x^2}\,dx\Big)\Big(\int_0^\infty e^{-y^2}\,dy\Big) = \int_0^\infty \Big(\int_0^\infty e^{-y^2}\,dy\Big)e^{-x^2}\,dx \\ &= \int_0^\infty \Big(\int_0^\infty e^{-x^2}e^{-y^2}\,dy\Big)\,dx = \int_0^\infty \Big(\int_0^\infty e^{-(x^2+y^2)}\,dy\Big)\,dx = \iint_R e^{-(x^2+y^2)}\,dA\,, \end{aligned}$$

where R is the first quadrant of the plane. In polar coordinate, the first quadrant can be expressed as $0 < r < \infty$ and $0 < \theta < \dfrac{\pi}{2}$; thus using the polar coordinate we find that

$$I^2 = \int_0^{\frac{\pi}{2}} \Big(\int_0^\infty e^{-r^2} r\,dr\Big)d\theta = \int_0^{\frac{\pi}{2}} \Big(-\frac{1}{2}e^{-r^2}\Big)\Big|_{r=0}^{r=\infty}\,d\theta = \frac{\pi}{4}\,.$$

By the fact that $I \geqslant 0$, we conclude that $I = \dfrac{\sqrt{\pi}}{2}$.

Example 14.29. In this example we reconsider the volume of the solid region Q in Example 14.16, where

$$Q = \big\{(x,y,z)\,\big|\,(x,y) \in R, x^2+y^2 \leqslant z \leqslant \sqrt{6-x^2-y^2}\big\}\,,$$

and R is a disk centered at the origin with radius $\sqrt{2}$.

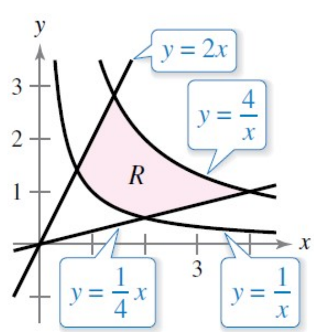
Using the cylindrical coordinate, the region Q can be expressed as

$$\big\{(r,\theta,z)\,\big|\,0 \leqslant r \leqslant \sqrt{2}, 0 \leqslant \theta \leqslant 2\pi, r^2 \leqslant z \leqslant \sqrt{6-r^2}\big\}\,.$$

Therefore, the volume of Q is given by

$$\begin{aligned} \iiint_Q dV &= \int_0^{2\pi}\Big[\int_0^{\sqrt{2}}\Big(\int_{r^2}^{\sqrt{6-r^2}} r\,dz\Big)dr\Big]d\theta = \int_0^{2\pi}\Big[\int_0^{\sqrt{2}} r\big(\sqrt{6-r^2}-r^2\big)\,dr\Big]d\theta \\ &= \int_0^{2\pi}\Big[-\frac{1}{3}(6-r^2)^{\frac{3}{2}} - \frac{1}{4}r^4\Big]\Big|_{r=0}^{r=\sqrt{2}}\,d\theta = \int_0^{2\pi}\big(-\frac{8}{3}-1+2\sqrt{6}\big)\,d\theta = 2\pi\big(2\sqrt{6}-\frac{11}{3}\big)\,. \end{aligned}$$

Example 14.30. Find the double integral $\displaystyle\iint_R e^{-\frac{xy}{2}}\,dA$, where R is the region given in the following figure.



Consider the following change of variables: $x = \sqrt{\frac{v}{u}}$ and $y = \sqrt{uv}$. In order to apply the change of variables formula to find the double integral, we need to know

1. What is the Jacobian of this change of variable?

2. What is the corresponding region of integration in the uv-plane?

We first note that for the change of variables to make sense, u, v have the same sign. W.L.O.G., we assume that the corresponding region in the uv-plane lies in the first quadrant. We compute the Jacobian and find that

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{1}{2}\sqrt{\frac{u}{v}} \cdot \frac{-v}{u^2} & \frac{1}{2}\sqrt{\frac{u}{v}} \cdot \frac{1}{u} \\ \frac{1}{2}\frac{v}{\sqrt{uv}} & \frac{1}{2}\frac{u}{\sqrt{uv}} \end{vmatrix} = \frac{1}{4} \cdot \frac{-1}{u} - \frac{1}{4} \cdot \frac{1}{u} = -\frac{1}{2u}\,.$$

Now we find the corresponding region R' in the uv-plane. The rule of thumb is that a one-to-one continuously differentiable function whose Jacobian does not vanish maps the boundary of a region to the boundary of its image. Therefore, the boundary of R' is given by $u = \frac{1}{2}$, $u = 2$ and $v = 1$, $v = 4$. Since the point (x, y) satisfying $xy = 2$ and $\frac{y}{x} = 1$ corresponds to $u = 1$ and $v = 2$, we find that $R' = \left[\frac{1}{2}, 2\right] \times [1, 4]$. Therefore, the change of variable formula implies that

$$\iint_R e^{-\frac{xy}{2}}\,dA = \iint_{[\frac{1}{2},2]\times[1,4]} e^{-\frac{v}{2}} \frac{1}{2u}\,dA' = \int_{\frac{1}{2}}^{2} \left(\int_1^4 \frac{e^{-\frac{v}{2}}}{2u}\,dv \right) du$$
$$= \int_{\frac{1}{2}}^{2} \left[\left(\frac{-e^{-\frac{v}{2}}}{u} \right)\Big|_{v=1}^{v=4} \right] du = \left(e^{-\frac{1}{2}} - e^{-2}\right) \int_{\frac{1}{2}}^{2} \frac{1}{u}\,du = 3\ln 2\left(e^{-\frac{1}{2}} - e^{-2}\right).$$

A more fundamental question is: why do we choose this change of coordinate? The general philosophy is to "straighten" the boundary so that in the new coordinate system the corresponding region becomes a region bounded by straight lines. Observing that the boundaries of the region R consists of four curves $\frac{y}{x} = \frac{1}{4}$, $\frac{y}{x} = 2$, $xy = 1$ and $xy = 4$, it is quite intuitive that we choose $u = \frac{y}{x}$ and $v = xy$ as our change of variables (in a reverse order). Solving for x, y in terms of u, v, we find that $x = \sqrt{\frac{v}{u}}$ and $y = \sqrt{uv}$.