微積分 MA1002-A 上課筆記（精簡版）

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Let Q be a bounded region in the space, and $f : Q \to \mathbb{R}$ be a non-negative function which described the point density of the region. We are interested in the mass of Q.

We start with the simple case that $Q = [a, b] \times [c, d] \times [r, s]$ is a cube. Let

$$\begin{aligned}
\mathcal{P}_x &= \{a = x_0 < x_1 < \cdots < x_n = b\}\,,\\
\mathcal{P}_y &= \{c = y_0 < y_1 < \cdots < y_m = d\}\,,\\
\mathcal{P}_z &= \{r = z_0 < z_1 < \cdots < z_p = s\}\,,
\end{aligned}$$

be partitions of $[a, b]$, $[c, d]$, $[r, s]$, respectively, and $\mathcal{P}$ be a collection of non-overlapping cubes given by

$$\mathcal{P} = \big\{R_{ijk}\,\big|\,R_{ijk} = [x_{i-1}, x_i] \times [y_{j-1}, y_j] \times [z_{k-1}, z_k], 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m, 1 \leqslant k \leqslant p\big\}\,.$$

Such a collection $\mathcal{P}$ is called a partition of Q, and the norm of $\mathcal{P}$ is the maximum of the length of the diagonals of all R_{ijk}; that is

$$\|\mathcal{P}\| = \max\Big\{\sqrt{(x_i - x_{i-1})^2 + (y_j - y_{j-1})^2 + (z_k - z_{k-1})^2}\,\Big|\,1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m, 1 \leqslant k \leqslant p\Big\}\,.$$

A Riemann sum of f for this partition $\mathcal{P}$ is given by

$$\sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{p} f(\xi_{ijk}, \eta_{ijk}, \zeta_{ijk})(x_i - x_{i-1})(y_j - y_{j-1})(z_k - z_{k-1})\,,$$

where $\big\{(\xi_{ijk}, \eta_{ijk}, \zeta_{ijk})\big\}_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m, 1\leqslant k\leqslant p}$ is a collection of points satisfying $(\xi_{ijk}, \eta_{ijk}, \zeta_{ijk}) \in Q_{ijk}$ for all $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$ and $1 \leqslant k \leqslant p$. The mass of Q then should be the "limit" of Riemann sums as $\|\mathcal{P}\|$ approaches zero. In general, we can remove the restrictions that f is non-negative on R and still consider the limit of the Riemann sums. We have the following

Theorem 14.14

Let $Q = [a, b] \times [c, d] \times [r, s]$ be a cube in the space, and $f : Q \to \mathbb{R}$ be a function. f is said to be Riemann integrable on Q if there exists a real number I such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\mathcal{P}$ is a partition of Q satisfying $\|\mathcal{P}\| < \delta$, then any Riemann sum of f for $\mathcal{P}$ belongs to $(I - \varepsilon, I + \epsilon)$. Such a number I (is unique if it exists and) is called the ***Riemann integral*** or ***triple integral of f on*** Q and is denoted by $\displaystyle\iiint_Q f(x, y, z)\,dV$.

For general bounded region Q in the space, let $r > 0$ be such that $Q \subseteq [-r, r]^3$, and we define $\displaystyle\iiint\limits_Q f(x,y,z)\,dV$ as $\displaystyle\iiint\limits_{[-r,r]^3} \widetilde{f}(x,y,z)\,dV$, where $\widetilde{f}$ is the zero extension of f given by

$$\widetilde{f}(x,y,z) = \begin{cases} f(x,y,z) & \text{if } (x,y,z) \in R\,, \\ 0 & \text{if } (x,y,z) \notin R\,. \end{cases}$$

Some of the properties of double integrals in Theorem 14.4 can be restated in terms of triple integrals.

1. $\displaystyle\iiint\limits_Q (cf)(x,y,z)\,dV = c\iiint\limits_Q f(x,y,z)\,dV$ for all Riemann integrable function f.
2. $\displaystyle\iiint\limits_Q (f+g)(x,y,z)\,dV = \iiint\limits_Q f(x,y,z)\,dV + \iiint\limits_Q g(x,y,z)\,dV$ for all Riemann integrable functions f, g.
3. $\displaystyle\iiint\limits_{Q_1\cup Q_2} f(x,y,z)\,dV = \iiint\limits_{Q_1} f(x,y,z)\,dV + \iiint\limits_{Q_2} f(x,y,z)\,dV$ for all "non-overlapping" solid regions Q_1 and Q_2 and Riemann integrable function f.

Similar to Fubini's Theorem for the evaluation of double integrals, we have the following

Theorem 14.15: Fubini's Theorem

Let Q be a region in the space, and $f : Q \to \mathbb{R}$ be continuous. If Q is given by $Q = \big\{(x,y,z)\,\big|\,(x,y) \in R, g_1(x,y) \leqslant z \leqslant g_2(x,y)\big\}$ for some region R in the xy-plane, then (f is Riemann integrable on Q and)

$$\iiint\limits_Q f(x,y,z)\,dV = \iint\limits_R \Big(\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\Big)\,dA\,.$$

In particular, if R is expressed by $R = \big\{(x,y)\,\big|\,a \leqslant x \leqslant b, h_1(x) \leqslant y \leqslant h_2(y)\big\}$, then

$$\iiint\limits_Q f(x,y,z)\,dV = \int_a^b \Big[\int_{h_1(x)}^{h_2(x)} \Big(\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)\,dz\Big)dy\Big]\,dx\,.$$

The integral which appears in the right-hand side of the last line of the theorem above is also an iterated integral.

Example 14.16. Find the volume of the region Q bounded below by the paraboloid $z = x^2 + y^2$ and above by the sphere $x^2 + y^2 + z^2 = 6$.

Suppose Q is a solid region in the space with uniform density 1 (or say, this region is occupied by water). Then the volume of Q is identical to the mass (in terms of its numerical value); thus we find that the volume of Q is given by $\iiint\limits_Q 1\,dV$. To apply the Fubini Theorem, we need to express Q as $\{(x,y,z)\,|\,(x,y)\in R, g_1(x,y)\leqslant z\leqslant g_2(x,y)\}$. Nevertheless, if R is the bounded region in the plane enclosed by the curve $(x^2+y^2)^2+x^2+y^2=6$ (which in fact gives $x^2+y^2=2$), then

$$Q=\big\{(x,y,z)\,\big|\,(x,y)\in R, x^2+y^2\leqslant z\leqslant\sqrt{6-x^2-y^2}\big\}$$

and the Fubini Theorem implies that

$$\text{the volume of } Q=\int_R\Big(\int_{x^2+y^2}^{\sqrt{6-x^2-y^2}} 1\,dz\Big)\,dA\,.$$

Solving for R, we find that $R=\big\{(x,y)\,\big|-\sqrt{2}\leqslant x\leqslant\sqrt{2}, -\sqrt{2-x^2}\leqslant y\leqslant\sqrt{2-x^2}\big\}$; thus by the Fubini Theorem we find that

$$\text{the volume of } Q=\int_{-\sqrt{2}}^{\sqrt{2}}\Big[\int_{-\sqrt{2-x^2}}^{\sqrt{2-x^2}}\Big(\int_{x^2+y^2}^{\sqrt{6-x^2-y^2}} 1\,dz\Big)dy\Big]dx\,.$$

Example 14.17. Evaluate $\displaystyle\int_0^{\sqrt{\pi/2}}\Big[\int_x^{\sqrt{\pi/2}}\Big(\int_1^3 \sin(y^2)\,dz\Big)dy\Big]dx$.

Let $R=\big\{(x,y)\,\big|\,0\leqslant x\leqslant\sqrt{\pi/2}, x\leqslant y\leqslant\sqrt{\pi/2}\big\}$, then the domain of integration is given by

$$Q=\big\{(x,y,z)\,\big|\,0\leqslant x\leqslant\sqrt{\pi/2}, x\leqslant y\leqslant\sqrt{\pi/2}, 1\leqslant z\leqslant 3\big\}$$

and the iterated integral given above is the triple integral $\iiint\limits_Q \sin(y^2)\,dV$.

Since R can also be expressed as $R=\big\{(x,y)\,\big|\,0\leqslant y\leqslant\sqrt{\pi/2}, 0\leqslant x\leqslant y\big\}$, by the Fubini Theorem we find that

$$\begin{aligned}\int_0^{\sqrt{\pi/2}}\Big[\int_x^{\sqrt{\pi/2}}\Big(\int_1^3 \sin(y^2)\,dz\Big)dy\Big]dx &= \iiint\limits_Q \sin(y^2)\,dV\\ &=\int_0^{\sqrt{\pi/2}}\Big[\int_0^y\Big(\int_1^3 \sin(y^2)\,dz\Big)dx\Big]dy=\int_0^{\sqrt{\pi/2}} 2y\sin(y^2)\,dy=-\cos(y^2)\Big|_{y=0}^{y=\sqrt{\pi/2}}=1\,.\end{aligned}$$

Example 14.18. Compute the iterated integrals

$$\int_0^6 \Big[\int_{\frac{z}{2}}^3 \Big(\int_{\frac{z}{2}}^y dx\Big) dy\Big] dz + \int_0^6 \Big[\int_3^{\frac{12-z}{2}} \Big(\int_{\frac{z}{2}}^{6-y} dx\Big) dy\Big] dz\,,$$

then write the sum above as a single iterated integral in the order $dydzdx$ and $dzdydx$.

We compute the two integrals above as follows:

$$\begin{aligned}\int_0^6 \Big[\int_{\frac{z}{2}}^3 \Big(\int_{\frac{z}{2}}^y dx\Big) dy\Big] dz &= \int_0^6 \Big[\int_{\frac{z}{2}}^3 \big(y-\frac{z}{2}\big)\,dy\Big] dz = \int_0^6 \Big(\frac{y^2-yz}{2}\Big|_{y=\frac{z}{2}}^{y=3}\Big)\,dz \\ &= \frac{1}{2}\int_0^6 \big(9-3z+\frac{z^2}{4}\big)\,dz = \frac{1}{2}\Big(9z-\frac{3z^2}{2}+\frac{z^3}{12}\Big)\Big|_{z=0}^{z=6} = 9\,,\end{aligned}$$

and

$$\begin{aligned}\int_0^6 \Big[\int_3^{\frac{12-z}{2}} \Big(\int_{\frac{z}{2}}^{6-y} dx\Big) dy\Big] dz &= \int_0^6 \Big[\int_3^{\frac{12-z}{2}} \big(6-y-\frac{z}{2}\big)\,dy\Big] dz \\ &= \frac{1}{2}\int_0^6 \big(12y-y^2-yz\big)\Big|_{y=3}^{y=\frac{12-z}{2}} dz \\ &= \frac{1}{2}\int_0^6 \Big[6(12-z)-\frac{144-24z+z^2}{4}-\frac{(12-z)z}{2}-36+9+3z\Big)\,dz \\ &= \frac{1}{2}\int_0^6 \Big(72-6z-36+6z-\frac{z^2}{4}-6z+\frac{z^2}{2}-27+3z\Big)\,dz \\ &= \frac{1}{2}\int_0^6 \big(9-3z+\frac{z^2}{4}\big)\,dz = \frac{1}{2}\Big(9z-\frac{3z^2}{2}+\frac{z^3}{12}\Big)\Big|_{z=0}^{z=6} = 9\,.\end{aligned}$$

Therefore, the sum of the two integrals is 18.

Let

$$\begin{aligned} Q_1 &= \Big\{(x,y,z)\,\Big|\, 0 \leqslant z \leqslant 6, \frac{z}{2} \leqslant y \leqslant 3, \frac{z}{2} \leqslant x \leqslant y\Big\}\,, \\ Q_2 &= \Big\{(x,y,z)\,\Big|\, 0 \leqslant z \leqslant 6, 3 \leqslant y \leqslant \frac{12-z}{2}, \frac{z}{2} \leqslant x \leqslant 6-y\Big\}\,. \end{aligned}$$

Then the Fubini Theorem implies that

$$\int_0^6 \Big[\int_{\frac{z}{2}}^3 \Big(\int_{\frac{z}{2}}^y dx\Big) dy\Big] dz = \iiint_{Q_1} dV\,, \qquad \int_0^6 \Big[\int_3^{\frac{12-z}{2}} \Big(\int_{\frac{z}{2}}^{6-y} dx\Big) dy\Big] dz = \iiint_{Q_2} dV\,.$$

Let $Q = Q_1 \cup Q_2$. Since Q_1 and Q_2 are non-overlapping solid regions (their intersection is a subset of the plane $y = 3$). Then

$$\iiint_{Q_1} dV + \iiint_{Q_2} dV = \iiint_{Q} dV\,.$$

1. Let R be the projection of Q onto the xz-plane. Then $R = \big\{(x,z)\,\big|\,0 \leqslant x \leqslant 3, 0 \leqslant z \leqslant 2x\big\}$ (where $z = 2x$ is the projection of the plane $x = \dfrac{z}{2}$ onto the xz-plane), and Q can also be expressed as

$$Q = \big\{(x,y,z)\,\big|\,(x,z) \in R, x \leqslant y \leqslant 6-x\big\}\,.$$

Therefore, the volume of Q is given by

$$\begin{aligned}\int_0^3 \Big[\int_0^{2x} \Big(\int_x^{6-x} dy\Big) dz\Big] dx &= \int_0^3 \Big[\int_0^{2x} (6-2x)\, dz\Big] dx \\ &= \int_0^3 2x(6-2x)\, dx = \Big(6x^2 - \frac{4x^3}{3}\Big)\Big|_{x=0}^{x=3} = 54 - 36 = 18\,.\end{aligned}$$

2. Let S be the projection of Q onto the xy-plane. Then $S = \big\{(x,y)\,\big|\,0 \leqslant x \leqslant 3, x \leqslant y \leqslant 6-x\big\}$, and Q can also be expressed as

$$Q = \big\{(x,y,z)\,\big|\,(x,y) \in S, 0 \leqslant z \leqslant 2x\big\}\,.$$

Therefore, the volume of Q is given by

$$\int_0^3 \Big[\int_x^{6-x} \Big(\int_0^{2x} dz\Big) dy\Big] dx = \int_0^3 \Big[\int_x^{6-x} 2x\, dy\Big]\, dx = \int_0^3 2x(6-2x)\, dx = 18\,.$$