微積分 MA1002-A 上課筆記（精簡版）

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Let $R = [a,b] \times [c,d]$ be a rectangle in the plane, and $f : R \to \mathbb{R}$ be a continuously differentiable function. We are interested in the area of the surface

$$S = \big\{(x,y,z) \,\big|\, (x,y) \in R\,, z = f(x,y)\big\}\,.$$

Let $\mathcal{P} = \big\{R_{ij} \,\big|\, 1 \leqslant i \leqslant n\,, 1 \leqslant j \leqslant m\big\}$ be a partition of R. Partition each rectangle $R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j]$ into two triangles Δ^1_{ij} and Δ^2_{ij}, where Δ^1_{ij} has vertices (x_{i-1}, y_{j-1}), (x_i, y_{j-1}), (x_{i-1}, y_j) and Δ^2_{ij} has vertices (x_i, y_j), (x_{i-1}, y_j), (x_i, y_{j-1}). Then intuitively, the area of the surface $f(\Delta^1_{ij})$ can be approximated by the area of the triangle T^1_{ij} with vertices $\big(x_{i-1}, y_{j-1}, f(x_{i-1}, y_{j-1})\big)$, $\big(x_i, y_{j-1}, f(x_i, y_{j-1})\big)$ and $\big(x_i, y_j, f(x_i, y_j)\big)$, while the area of the surface $f(\Delta^2_{ij})$ can be approximated by the area of the triangle T^2_{ij} with vertices $\big(x_i, y_j, f(x_i, y_j)\big)$, $\big(x_{i-1}, y_j, f(x_{i-1}, y_j)\big)$ and $\big(x_i, y_{j-1}, f(x_i, y_{j-1})\big)$. Therefore, the area of the surface $f(R_{ij})$ can be approximated by the sum of area of triangles T^1_{ij} and T^2_{ij}, and the area of the surface S can be approximated by the sum of the area of the triangles T^1_{ij} and T^2_{ij}, where is sum is taken over all $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$.

Now we compute the area of the triangles T^1_{ij} and T^2_{ij}. We remark that for a triangle T with vertices P_1, P_2, P_3, letting $\boldsymbol{u} = \overrightarrow{P_1P_2} = P_2 - P_1$ and $\boldsymbol{v} = \overrightarrow{P_1P_3} = P_3 - P_1$, the area of T can be computed by $\frac{1}{2}\|\boldsymbol{u} \times \boldsymbol{v}\|$. Therefore, the area of T^1_{ij} is given by

$$\begin{aligned}|T^1_{ij}| = \frac{1}{2}\Big\|\big(x_i - x_{i-1}, 0, f(x_i, y_{j-1}) - f(x_{i-1}, y_{j-1})\big) \times \\ \times \big(0, y_j - y_{j-1}, f(x_{i-1}, y_j) - f(x_{i-1}, y_{j-1})\big)\Big\|\,.\end{aligned}$$

By the mean value theorem, there exist $\xi^*_i \in (x_{i-1}, x_i)$ and $\eta^*_j \in (y_{j-1}, y_j)$ such that

$$\begin{aligned}f(x_i, y_{j-1}) - f(x_{i-1}, y_{j-1}) &= f_x(\xi^*_i, y_{j-1})(x_i - x_{i-1})\,,\\ f(x_{i-1}, y_j) - f(x_{i-1}, y_{j-1}) &= f_y(x_{i-1}, \eta^*_j)(y_j - y_{j-1})\,;\end{aligned}$$

thus we obtain that

$$\begin{aligned}|T^1_{ij}| &= \frac{1}{2}\big\|\big(1, 0, f_x(\xi^*_i, y_{j-1})\big) \times \big(0, 1, f_y(x_{i-1}, \eta^*_j)\big)\big\|(x_i - x_{i-1})(y_j - y_{j-1})\\ &= \frac{1}{2}\big\|\big(-f_x(\xi^*_i, y_{j-1}), -f_y(x_{i-1}, \eta^*_j), 1\big)\big\|(x_i - x_{i-1})(y_j - y_{j-1})\\ &= \frac{1}{2}\sqrt{1 + f_x(\xi^*_i, y_{j-1})^2 + f_y(x_{i-1}, \eta^*_j)^2}(x_i - x_{i-1})(y_j - y_{j-1})\,.\end{aligned}$$

Similarly, there exist $\xi^{**}_i \in (x_{i-1}, x_i)$ and $\eta^{**}_j \in (y_{j-1}, y_j)$ such that the area of the triangle T^2_{ij} is given by

$$|T^2_{ij}| = \frac{1}{2}\sqrt{1 + f_x(\xi^{**}_i, y_j)^2 + f_y(x_i, \eta^{**}_j)^2}(x_i - x_{i-1})(y_j - y_{j-1})\,.$$

Let $M = \max\limits_{(x,y)\in R} \big(|f_x(x,y)| + |f_y(x,y)|\big)$, $|R| = (b-a)(d-c)$, and $\varepsilon > 0$ be a given (but arbitrary) number. Suppose that

$$\big|f_x(\alpha,\beta) - f_x(\xi,\eta)\big| + \big|f_y(\alpha,\beta) - f_y(\xi,\eta)\big| < \frac{\varepsilon}{2|R|(1+M)} \qquad \forall\,(\alpha,\beta),(\xi,\eta) \in R_{ij}\,. \quad (14.3.1)$$

Then

$$\begin{aligned}
&\Big|\sqrt{1+f_x(\alpha,\beta)^2+f_y(\alpha^*,\beta^*)^2} - \sqrt{1+f_x(\xi,\eta)^2+f_y(\xi,\eta)^2}\Big| \\
&\quad = \left|\frac{f_x(\alpha,\beta)^2+f_y(\alpha^*,\beta^*)^2 - f_x(\xi,\eta)^2 - f_y(\xi,\eta)^2}{\sqrt{1+f_x(\alpha,\beta)^2+f_y(\alpha^*,\beta^*)^2} + \sqrt{1+f_x(\xi,\eta)^2+f_y(\xi,\eta)^2}}\right| \\
&\quad \leqslant \frac{1}{2}\Big[\big|f_x(\alpha,\beta) - f_x(\xi,\eta)\big|\big|f_x(\alpha,\beta) + f_x(\xi,\eta)\big| \\
&\qquad + \big|f_y(\alpha^*,\beta^*) - f_y(\xi,\eta)\big|\big|f_y(\alpha^*,\beta^*) + f_y(\xi,\eta)\big|\Big] \\
&\quad \leqslant \frac{2M}{2}\Big[\big|f_x(\alpha,\beta) - f_x(\xi,\eta)\big| + \big|f_y(\alpha^*,\beta^*) - f_y(\xi,\eta)\big|\Big] \leqslant \frac{M\varepsilon}{2|R|(1+M)} < \frac{\varepsilon}{2|R|}\,.
\end{aligned}$$

Therefore, if (14.3.1) holds for all $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$, then for $(\xi_{ij},\eta_{ij}) \in R_{ij}$, we have

$$\begin{aligned}
&\Big||T^1_{ij}| + |T^2_{ij}| - \sqrt{1+f_x(\xi_{ij},\eta_{ij})^2+f_y(\xi_{ij},\eta_{ij})^2}(x_i-x_{i-1})(y_j-y_{j-1})\Big| \\
&\quad \leqslant \Big|\frac{1}{2}\sqrt{1+f_x(\xi^*_i,y_{j-1})^2+f_y(x_{i-1},\eta^*_j)^2} + \frac{1}{2}\sqrt{1+f_x(\xi^{**}_i,y_j)^2+f_y(x_i,\eta^{**}_j)^2} \\
&\qquad - \sqrt{1+f_x(\xi_{ij},\eta_{ij})^2+f_y(\xi_{ij},\eta_{ij})^2}\Big|(x_i-x_{i-1})(y_j-y_{j-1}) \\
&\quad \leqslant \frac{\varepsilon}{2|R|}(x_i-x_{i-1})(y_j-y_{j-1})\,;
\end{aligned}$$

thus if (14.3.1) holds for all $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$, then for $(\xi_{ij},\eta_{ij}) \in R_{ij}$,

$$\begin{aligned}
&\left|\sum_{i=1}^n\sum_{j=1}^m \big(|T^1_{ij}| + |T^2_{ij}|\big) - \sum_{i=1}^n\sum_{j=1}^m \sqrt{1+f_x(\xi_{ij},\eta_{ij})^2+f_y(\xi_{ij},\eta_{ij})^2}(x_i-x_{i-1})(y_j-y_{j-1})\right| \\
&\quad \leqslant \sum_{i=1}^n\sum_{j=1}^m \Big||T^1_{ij}| + |T^2_{ij}| - \sqrt{1+f_x(\xi_{ij},\eta_{ij})^2+f_y(\xi_{ij},\eta_{ij})^2}(x_i-x_{i-1})(y_j-y_{j-1})\Big| \\
&\quad \leqslant \sum_{i=1}^n\sum_{j=1}^m \frac{\varepsilon}{2|R|}(x_i-x_{i-1})(y_j-y_{j-1}) = \frac{\varepsilon}{2}\,.
\end{aligned}$$

Finally, we state as a fact that there exists $\delta_1 > 0$ such that (14.3.1) holds as long as $\|\mathcal{P}\| < \delta_1$. This property is called the ***uniform continuity*** of continuous functions on closed and bounded sets.

On the other hand, since the function $z = \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}$ is continuous on R (and R has area), it is Riemann integrable on R. Let

$$\mathrm{I} = \iint\limits_R \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\, dA\,.$$

Then there exists $\delta_2 > 0$ such that if $\mathcal{P}$ is a partition of R satisfying $\|\mathcal{P}\| < \delta_2$, then any Riemann sum of f for the partition $\mathcal{P}$ belongs to $\big(\mathrm{I} - \frac{\varepsilon}{2}, \mathrm{I} + \frac{\varepsilon}{2}\big)$. Therefore,

$$\Big|\sum_{i=1}^{n}\sum_{j=1}^{m} \sqrt{1 + f_x(\xi_{ij},\eta_{ij})^2 + f_y(\xi_{ij},\eta_{ij})^2}(x_i - x_{i-1})(y_j - y_{j-1}) - \mathrm{I}\Big| < \frac{\varepsilon}{2}\,.$$

Let $\delta = \min\{\delta_1, \delta_2\}$. Then $\delta > 0$, and if $\mathcal{P} = \big\{R_{ij} \,\big|\, R_{ij} = [x_{i-1}, x_i] \times [y_{j-1}, y_j], 1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m\big\}$ is a partition of R satisfying $\|\mathcal{P}\| < \delta$, then by choosing a collection of points $\big\{(\xi_{ij}, \eta_{ij})\big\}_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m}$ such that $(\xi_{ij}, \eta_{ij}) \in R_{ij}$, we conclude that

$$\begin{aligned}
&\Big|\sum_{i=1}^{n}\sum_{j=1}^{m} \big(|T_{ij}^1| + |T_{ij}^2|\big) - \mathrm{I}\Big| \\
&\quad \leqslant \Big|\sum_{i=1}^{n}\sum_{j=1}^{m} \big(|T_{ij}^1| + |T_{ij}^2|\big) - \sum_{i=1}^{n}\sum_{j=1}^{m} \sqrt{1 + f_x(\xi_{ij},\eta_{ij})^2 + f_y(\xi_{ij},\eta_{ij})^2}(x_i - x_{i-1})(y_j - y_{j-1})\Big| \\
&\qquad + \Big|\sum_{i=1}^{n}\sum_{j=1}^{m} \sqrt{1 + f_x(\xi_{ij},\eta_{ij})^2 + f_y(\xi_{ij},\eta_{ij})^2}(x_i - x_{i-1})(y_j - y_{j-1}) - \mathrm{I}\Big| < \varepsilon\,.
\end{aligned}$$

This means that the approximation of the area of the surface S can be made arbitrarily closed to I; thus the area of the surface S must be I. In general, we have the following

Theorem 14.11

Let R be a closed region in the plane, and $f : R \to \mathbb{R}$ be a continuously differentiable function. Then the area of the surface $S = \big\{(x,y,z) \,\big|\, (x,y) \in R\,, z = f(x,y)\big\}$ is given by

$$\iint\limits_R \sqrt{1 + \|(\nabla f)(x,y)\|^2}\, dA = \int_R \sqrt{1 + f_x(x,y)^2 + f_y(x,y)^2}\, dA\,.$$

Example 14.12. In this example we consider the surface area of the upper hemi-sphere $z = \sqrt{r^2 - x^2 - y^2}$ that lies above the disk $R = \big\{(x,y) \,\big|\, x^2 + y^2 \leqslant \sigma^2\big\}$, where $0 < \sigma \leqslant r$. Since R can also be expressed by

$$R = \big\{(x,y) \,\big|\, -r\sigma \leqslant x \leqslant \sigma, -\sqrt{\sigma^2 - x^2} \leqslant y \leqslant \sqrt{\sigma^2 - x^2}\big\}\,,$$

the computations from the previous example, as well as the Fubini Theorem, implies that the surface area of interest is given by

$$\iint\limits_R \frac{r}{\sqrt{r^2-x^2-y^2}}\,dA = r\int_{-\sigma}^{\sigma}\Big(\int_{-\sqrt{\sigma^2-x^2}}^{\sqrt{\sigma^2-x^2}} \frac{1}{\sqrt{r^2-x^2-y^2}}\,dy\Big)dx\,.$$

Using $\displaystyle\int \frac{1}{\sqrt{a^2-u^2}}\,du = \arcsin\frac{u}{a} + C$, we find that

$$\begin{aligned}
\int_{-\sigma}^{\sigma}\Big(\int_{-\sqrt{\sigma^2-x^2}}^{\sqrt{\sigma^2-x^2}} \frac{1}{\sqrt{r^2-x^2-y^2}}\,dy\Big)dx &= \int_{-\sigma}^{\sigma}\bigg(\arcsin\frac{y}{\sqrt{r^2-x^2}}\bigg|_{y=-\sqrt{\sigma^2-x^2}}^{y=\sqrt{\sigma^2-x^2}}\bigg)dx\\
&= 2\int_{-\sigma}^{\sigma} \arcsin\frac{\sqrt{\sigma^2-x^2}}{\sqrt{r^2-x^2}}\,dx = 2\int_{-\sigma}^{\sigma}\arctan\frac{\sqrt{\sigma^2-x^2}}{\sqrt{r^2-\sigma^2}}\,dx\\
&= 2\Big[x\arctan\frac{\sqrt{\sigma^2-x^2}}{\sqrt{r^2-\sigma^2}}\bigg|_{x=-\sigma}^{x=\sigma} - \int_{-\sigma}^{\sigma} x\frac{d}{dx}\arctan\frac{\sqrt{\sigma^2-x^2}}{\sqrt{r^2-\sigma^2}}\,dx\Big]\\
&= -2\int_{-\sigma}^{\sigma}\frac{x\cdot\frac{1}{\sqrt{r^2-\sigma^2}}\frac{-x}{\sqrt{\sigma^2-x^2}}}{1+\frac{\sigma^2-x^2}{r^2-\sigma^2}}\,dx = 2\sqrt{r^2-\sigma^2}\int_{-\sigma}^{\sigma}\frac{x^2-r^2+r^2}{(r^2-x^2)\sqrt{\sigma^2-x^2}}\,dx\\
&= -2\sqrt{r^2-\sigma^2}\pi + 2\sqrt{r^2-\sigma^2}\int_{-\sigma}^{\sigma}\frac{r^2}{(r^2-x^2)\sqrt{\sigma^2-x^2}}\,dx\,.
\end{aligned}$$

Using the substitution $x=\sigma\sin\frac{\theta}{2}$, we find that

$$\begin{aligned}
\int_{-\sigma}^{\sigma}\frac{r^2}{(r^2-x^2)\sqrt{\sigma^2-x^2}}\,dx &= \int_{-\pi}^{\pi}\frac{r^2}{2(r^2-\sigma^2\sin^2\frac{\theta}{2})}\,d\theta = \int_{-\pi}^{\pi}\frac{r^2}{2r^2-\sigma^2(1-\cos\theta)}\,d\theta\\
&= r^2\int_{-\pi}^{\pi}\frac{1}{(2r^2-\sigma^2)+\sigma^2\cos\theta}\,d\theta\,.
\end{aligned}$$

and further substitution $\tan\frac{\theta}{2}=t$ implies that

$$\begin{aligned}
\int_{-\sigma}^{\sigma}\frac{r^2}{(r^2-x^2)\sqrt{\sigma^2-x^2}}\,dx &= \int_{-\infty}^{\infty}\frac{r^2}{(2r^2-\sigma^2)+\sigma^2\frac{1-t^2}{1+t^2}}\frac{2dt}{1+t^2}\\
&= \int_{-\infty}^{\infty}\frac{2r^2}{2r^2(1+t^2)-\sigma^2(1+t^2)+\sigma^2(1-t^2)}\,dt\\
&= \int_{-\infty}^{\infty}\frac{r^2}{r^2+(r^2-\sigma^2)t^2}\,dt\\
&= \frac{r}{\sqrt{r^2-\sigma^2}}\arctan\Big(\frac{\sqrt{r^2-\sigma^2}}{r}t\Big)\bigg|_{t=-\infty}^{\infty} = \frac{\pi r}{\sqrt{r^2-\sigma^2}}\,.
\end{aligned}$$

Therefore, the surface area of interest is given by

$$\iint_R \frac{r}{\sqrt{r^2 - x^2 - y^2}}\, dA = 2r\sqrt{r^2-\sigma^2}\left[-\pi + \frac{\pi r}{\sqrt{r^2-\sigma^2}}\right] = 2\pi r\left(r - \sqrt{r^2-\sigma^2}\right).$$

We also note that the surface area that we obtain approaches $2\pi r^2$ as $\sigma \to r$. This value $2\pi r^2$ is exactly the surface area of the upper hemi-sphere with radius r.