微積分 MA1002-A 上課筆記（精簡版）

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Suppose that $R = [a,b] \times [c,b] = \big\{(x,y) \,\big|\, a \leqslant x \leqslant b, c \leqslant y \leqslant d\big\}$ is a rectangular region in the plane, $f : R \to \mathbb{R}$ is a non-negative continuous function. Let $\mathcal{P}_x = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ and $\mathcal{P}_y = \{c = y_0 < y_1 < \cdots < y_m = d\}$ be partitions of $[a,b]$ and $[c,d]$, respectively, and R_{ij} denote the rectangle $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$. The collection of rectangles $\mathcal{P} = \{R_k \,|\, 1 \leqslant k \leqslant nm\}$ is called a partition of R. A Riemann sum of f for $\mathcal{P}$ is of the form

$$\sum_{i=1}^{n} \sum_{j=1}^{m} f(\xi_{ij}, \eta_{ij}) A_{ij} \,,$$

where $\big\{(\xi_{ij}, \eta_{ij})\big\}_{1 \leqslant i \leqslant n, 1 \leqslant j \leqslant m}$ is a collection of point in R such that $(\xi_{ij}, \eta_{ij}) \in R_{ij}$, and $A_{ij} = (x_i - x_{i-1})(y_j - y_{j-1})$ is the area of the rectangle R_{ij}. Define the norm of $\mathcal{P}$, denoted by $\|\mathcal{P}\|$, as the maximum length of the diagonal of R_{ij}.

Definition 14.1

Let $R = [a,b] \times [c,d]$ be a rectangle in the plane, and $f : R \to \mathbb{R}$ be a function. f is said to be Riemann integrable on R if there exists a real number V such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\mathcal{P}$ is partition of R satisfying $\|\mathcal{P}\| < \delta$, then any Riemann sums of f for the partition $\mathcal{P}$ belongs to the interval $(V - \varepsilon, V + \varepsilon)$. Such a number V (is unique if it exists and) is called the ***Riemann integral*** or ***double integral of*** f ***on*** R and is denoted by $\displaystyle\iint_R f(x,y)\,dA$.

For general bounded region R in the plane, let $r > 0$ be such that $R \subseteq [-r,r]^2$, and we define $\displaystyle\iint_R f(x,y)\,dA$ as $\displaystyle\iint_{[-r,r]^2} \widetilde{f}(x,y)\,dA$, where $\widetilde{f}$ is the zero extension of f.

Theorem 14.7: Fubini's Theorem

Let R be a region in the plane, and $f : R \to \mathbb{R}$ be continuous (but no necessary non-negative).

1. If R is given by $R = \big\{(x,y) \,\big|\, a \leqslant x \leqslant b, g_1(x) \leqslant y \leqslant g_2(x)\big\}$, then
$$\iint_R f(x,y)\,dA = \int_a^b \Big(\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy \Big)\,dx \,.$$
2. If R is given by $R = \big\{(x,y) \,\big|\, c \leqslant y \leqslant d, g_1(x) \leqslant y \leqslant g_2(x)\big\}$, then
$$\iint_R f(x,y)\,dA = \int_c^d \Big(\int_{h_1(y)}^{h_2(y)} f(x,y)\,dx \Big)\,dy \,.$$

Example 14.8. Find the volume of the solid region bounded by the paraboloid $z = 4 - x^2 - 2y^2$ and the xy-plane. By the definition of double integrals, the volume of this solid is given by $\iint_R (4 - x^2 - 2y^2)\, dA$, where R is the region $\{(x, y) \,|\, x^2 + 2y^2 \leqslant 4\}$. Writing R as

$$R = \Big\{(x, y) \,\Big|\, -2 \leqslant x \leqslant 2\,, -\sqrt{\frac{4-x^2}{2}} \leqslant y \leqslant \sqrt{\frac{4-x^2}{2}}\Big\}$$

or

$$R = \big\{(x, y) \,|\, -\sqrt{2} \leqslant y \leqslant \sqrt{2}\,, -\sqrt{4-2y^2} \leqslant x \leqslant \sqrt{4-2y^2}\big\}\,,$$

the Fubini Theorem then implies that

$$\begin{aligned}\iint_R (4 - x^2 - 2y^2)\, dA &= \int_{-2}^{2} \Big(\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (4 - x^2 - 2y^2)\, dy\Big)\, dx \\ &= \int_{-\sqrt{2}}^{\sqrt{2}} \Big(\int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} (4 - x^2 - 2y^2)\, dx\Big)\, dy\,.\end{aligned}$$

1. Integrating in y first then integrating in x: for fixed $x \in [-2, 2]$,

$$\begin{aligned}\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (4 - x^2 - 2y^2)\, dy &= \int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} (4 - x^2)\, dy - 2\int_{-\sqrt{\frac{4-x^2}{2}}}^{\sqrt{\frac{4-x^2}{2}}} y^2\, dy \\ &= \sqrt{2}(4 - x^2)^{\frac{3}{2}} - \frac{4}{3}\Big(\sqrt{\frac{4-x^2}{2}}\Big)^3 = \frac{2\sqrt{2}}{3}(4 - x^2)^{\frac{3}{2}}\,.\end{aligned}$$

Therefore, by the substitution $x = 2\sin\theta$ (so that $dx = 2\cos\theta\, d\theta$),

$$\begin{aligned}\iint_R (4 - x^2 - 2y^2)\, dA &= \frac{2\sqrt{2}}{3}\int_{-2}^{2} (4 - x^2)^{\frac{3}{2}}\, dx = \frac{2\sqrt{2}}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 8\cos^3\theta \cdot 2\cos\theta d\theta \\ &= \frac{32\sqrt{2}}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta\, d\theta = \frac{64\sqrt{2}}{3}\int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta \\ &= \frac{64\sqrt{2}}{3}\int_0^{\frac{\pi}{2}} \Big(\frac{1 + \cos 2\theta}{2}\Big)^2 d\theta \\ &= \frac{16\sqrt{2}}{3}\int_0^{\frac{\pi}{2}} \Big(1 + 2\cos 2\theta + \frac{1 + \cos 4\theta}{2}\Big)\, d\theta \\ &= \frac{16\sqrt{2}}{3}\Big[\frac{3}{2} \cdot \frac{\pi}{2} + \sin\Big(2 \cdot \frac{\pi}{2}\Big) + \frac{1}{8}\sin\Big(4 \cdot \frac{\pi}{2}\Big)\Big] = 4\sqrt{2}\pi\,.\end{aligned}$$

2. Integrating in x first then integrating in y: for fixed $y \in [-\sqrt{2}, \sqrt{2}]$,

$$\int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} (4 - x^2 - 2y^2)\, dx = \int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} (4 - 2y^2)\, dx - \int_{-\sqrt{4-2y^2}}^{\sqrt{4-2y^2}} x^2\, dx$$
$$= 2(4-2y^2)^{\frac{3}{2}} - \frac{2}{3}(4-2y^2)^{\frac{3}{2}} = \frac{4}{3}(4-2y^2)^{\frac{3}{2}}\,;$$

thus by the substitution of variable $y = \sqrt{2}\sin\theta$ (so that $dy = \sqrt{2}\cos\theta\, d\theta$),

$$\iint_R (4 - x^2 - 2y^2)\, dA = \frac{4}{3}\int_{-\sqrt{2}}^{\sqrt{2}} (4-2y^2)^{\frac{3}{2}}\, dy = \frac{4}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 8\cos^3\theta \cdot \sqrt{2}\cos\theta\, d\theta$$
$$= \frac{32\sqrt{2}}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \cos^4\theta\, d\theta = \frac{64\sqrt{2}}{3}\int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta = 4\sqrt{2}\pi\,.$$

Example 14.9. Find the volume of the solid region bounded above by the paraboloid $z = 1 - x^2 - y^2$ and below by the plane $z = 1 - y$.

Let R be the region in the plane whose boundary points (x, y) satisfies $1 - x^2 - y^2 = 1 - y$ or equivalently, $x^2 + y^2 - y = 0$. Then the volume of the solid described above is given by $\iint_R \big[(1-x^2-y^2) - (1-y)\big]\, dA$. Note that the region R is a disk centered at $\big(0, \frac{1}{2}\big)$ with radius $\frac{1}{2}$ and can be written as

$$R = \big\{(x,y)\,\big|\, 0 \leqslant y \leqslant 1\,, -\sqrt{y-y^2} \leqslant x \leqslant \sqrt{y-y^2}\big\}\,.$$

Therefore,

$$\iint_R \big[(1-x^2-y^2) - (1-y)\big]\, dA = \int_0^1 \Big(\int_{-\sqrt{y-y^2}}^{\sqrt{y-y^2}} (y - x^2 - y^2)\, dx\Big) dy$$
$$= \int_0^1 \Big(2(y-y^2)^{\frac{3}{2}} - \frac{2}{3}(y-y^2)^{\frac{3}{2}}\Big)\, dy = \frac{4}{3}\int_0^1 (y-y^2)^{\frac{3}{2}}\, dy = \frac{4}{3}\int_0^1 \Big[\frac{1}{4} - \big(y - \frac{1}{2}\big)^2\Big]^{\frac{3}{2}}\, dy\,.$$

Making the substitution of variable $y - \frac{1}{2} = \frac{1}{2}\sin\theta$ $\big($so that $dy = \frac{1}{2}\cos\theta\, d\theta\big)$,

$$\iint_R \big[(1-x^2-y^2) - (1-y)\big]\, dA = \frac{4}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\cos^3\theta}{8} \cdot \frac{1}{2}\cos\theta\, d\theta = \frac{1}{6}\int_0^{\frac{\pi}{2}} \cos^4\theta\, d\theta = \frac{\pi}{32}\,.$$

Example 14.10. Find the iterated integral $\int_0^1 \Big(\int_y^1 e^{-x^2}\, dx\Big) dy$.

Let $R = \big\{(x,y)\,\big|\,0 \leqslant y \leqslant 1\,, y \leqslant x \leqslant 1\big\}$. Since R can also be expressed as $R = \big\{(x,y)\,\big|\,0 \leqslant x \leqslant 1\,, 0 \leqslant y \leqslant x\big\}$, by the Fubini Theorem we find that

$$\int_0^1 \Big(\int_y^1 e^{-x^2}\,dx\Big)\,dy = \iint\limits_R e^{-x^2}\,dA = \int_0^1 \Big(\int_0^x e^{-x^2}\,dy\Big)dx = \int_0^1 xe^{-x^2}\,dx$$
$$= -\frac{1}{2}e^{-x^2}\Big|_{x=0}^{x=1} = \frac{1}{2}\big(1-e^{-1}\big)\,.$$

14.3 Surface Area

Let $R = [a,b]\times[c,d]$ be a rectangle in the plane, and $f : R \to \mathbb{R}$ be a continuously differentiable function. We are interested in the area of the surface

$$S = \big\{(x,y,z)\,\big|\,(x,y)\in R\,, z = f(x,y)\big\}\,.$$

Let $\mathcal{P} = \big\{R_{ij}\,\big|\,1 \leqslant i \leqslant n\,, 1 \leqslant j \leqslant m\big\}$ be a partition of R. Partition each rectangle $R_{ij} = [x_{i-1},x_i]\times[y_{j-1},y_j]$ into two triangles Δ_{ij}^1 and Δ_{ij}^2, where Δ_{ij}^1 has vertices (x_{i-1},y_{j-1}), (x_i,y_{j-1}), (x_{i-1},y_j) and Δ_{ij}^2 has vertices (x_i,y_j), (x_{i-1},y_j), (x_i,y_{j-1}). Then intuitively, the area of the surface $f(\Delta_{ij}^1)$ can be approximated by the area of the triangle T_{ij}^1 with vertices $\big(x_{i-1},y_{j-1},f(x_{i-1},y_{j-1})\big)$, $\big(x_i,y_{j-1},f(x_i,y_{j-1})\big)$ and $\big(x_i,y_j,f(x_i,y_j)\big)$, while the area of the surface $f(\Delta_{ij}^2)$ can be approximated by the area of the triangle T_{ij}^2 with vertices $\big(x_i,y_j,f(x_i,y_j)\big)$, $\big(x_{i-1},y_j,f(x_{i-1},y_j)\big)$ and $\big(x_i,y_{j-1},f(x_i,y_{j-1})\big)$. Therefore, the area of the surface $f(R_{ij})$ can be approximated by the sum of area of triangles T_{ij}^1 and T_{ij}^2, and the area of the surface S can be approximated by the sum of the area of the triangles T_{ij}^1 and T_{ij}^2, where is sum is taken over all $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$.