微積分 MA1002-A 上課筆記（精簡版）

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Chapter 14

Multiple Integration

14.1 Double Integrals and Volume

Let R be a closed and bounded region in the plane, and $f : R \to \mathbb{R}$ be a non-negative continuous function. We are interested in the volume of the solid in the space

$$\mathrm{D} = \big\{(x, y, z) \,\big|\, (x, y) \in R\,, 0 \leqslant z \leqslant f(x, y)\big\}\,.$$

First we assume that $R = [a, b] \times [c, b] = \big\{(x, y) \,\big|\, a \leqslant x \leqslant b, c \leqslant y \leqslant d\big\}$ be a rectangle. Let $\mathcal{P}_x = \{a = x_0 < x_1 < x_2 < \cdots < x_n = b\}$ and $\mathcal{P}_y = \{c = y_0 < y_1 < \cdots < y_m = d\}$ be partitions of $[a, b]$ and $[c, d]$, respectively, R_{ij} denote the rectangle $[x_{i-1}, x_i] \times [y_{j-1}, y_j]$, and $\big\{(\alpha_i, \beta_j)\big\}_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m}$ be a collection of points such that $\alpha_i \in [x_{i-1}, x_i]$ and $\beta_j \in [y_{j-1}, y_j]$. Then as before, we consider an approximation of the volume of D given by

$$\sum_{i=1}^{n}\sum_{j=1}^{m} f(\alpha_i, \beta_j)(x_i - x_{i-1})(y_j - y_{j-1})\,.$$

Then the limit of the sum above, as $\|\mathcal{P}_x\|$, $\|\mathcal{P}_y\|$ approaches zero, is the volume of D. The collection of rectangles $\mathcal{P} = \{R_{ij}\}_{1\leqslant i\leqslant n, 1\leqslant j\leqslant m}$ is called a partition of R.

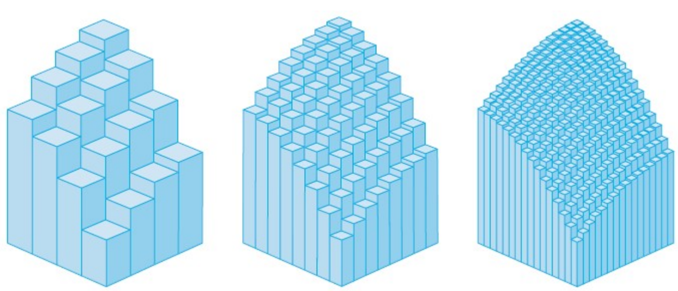

Figure 14.1: The volume of D can be obtained by making $\|\mathcal{P}_x\|, \|P_y\| \to 0$.

In general, by relabeling the rectangles as $R_1, R_2, \cdots, R_{nm}$ (thus $\mathcal{P} = \{R_k \,|\, 1 \leqslant k \leqslant nm\}$), and letting $\big\{(\xi_k, \eta_k)\big\}_{k=1}^{nm}$ be a collection of point in R such that $(\xi_k, \eta_k) \in R_k$, we can consider an approximation of the volume of the solid given by

$$\sum_{k=1}^{n} f(\xi_k, \eta_k) A_k\,,$$

where A_k is the area of the rectangle R_k. The sum above is called a ***Riemann sum of*** f ***for partition*** $\mathcal{P}$. With $\|\mathcal{P}\|$, called the norm of $\mathcal{P}$, denoting the maximum length of the diagonal of R_k; that is,

$$\|\mathcal{P}\| = \max\big\{\ell_k \,\big|\, \ell_k \text{ is the length of the diagonal of } R_k, 1 \leqslant k \leqslant nm\big\}\,,$$

then the volume of D is the "limit"

$$\lim_{\|\mathcal{P}\| \to 0} \sum_{k=1}^{n} f(\xi_k, \eta_k) A_k$$

as long as "the limit exists". Similar to the discussion of the limit of Riemann sums in the case of functions of one variable, we can remove the restrictions that f is continuous and non-negative on R and still consider the limit of the Riemann sums. We have the following

Definition 14.1

Let $R = [a, b] \times [c, d]$ be a rectangle in the plane, and $f : R \to \mathbb{R}$ be a function. f is said to be Riemann integrable on R if there exists a real number V such that for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $\mathcal{P}$ is partition of R satisfying $\|\mathcal{P}\| < \delta$, then any Riemann sums for the partition $\mathcal{P}$ belongs to the interval $(V - \varepsilon, V + \varepsilon)$. Such a number V (is unique if it exists and) is called the ***Riemann integral*** or ***double integral of*** f ***on*** R and is denoted by $\displaystyle\iint\limits_R f(x, y)\, dA$.