微積分 MA1002-A 上課筆記（精簡版）

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Theorem 13.37

Let $z = f(x, y)$ be a differentiable function (of x and y). If $x = g(t)$ and $y = h(t)$ are differentiable functions (of t), then $z(t) = f\big(x(t), y(t)\big)$ is differentiable and

$$z'(t) = f_x\big(x(t), y(t)\big)x'(t) + f_y\big(x(t), y(t)\big)y'(t)\,.$$

Corollary 13.38

Let $z = f(x, y)$ be a differentiable function (of x and y).

1. If $x = u(s,t)$ and $y = v(s,t)$ are such that $\dfrac{\partial u}{\partial s}$ and $\dfrac{\partial v}{\partial s}$ exist, then the first partial derivative $\dfrac{\partial z}{\partial s}$ of the function $z(s,t) = f\big(u(s,t), v(s,t)\big)$ exists and

$$z_s(s,t) = f_x\big(u(s,t), v(s,t)\big)u_s(s,t) + f_y\big(u(s,t), v(s,t)\big)v_s(s,t)\,.$$

2. If $x = u(s,t)$ and $y = v(s,t)$ are such that $\dfrac{\partial u}{\partial t}$ and $\dfrac{\partial v}{\partial t}$ exist, then the first partial derivative $\dfrac{\partial z}{\partial t}$ of the function $z(s,t) = f\big(u(s,t), v(s,t)\big)$ exists and

$$z_t(s,t) = f_x\big(u(s,t), v(s,t)\big)u_t(s,t) + f_y\big(u(s,t), v(s,t)\big)v_t(s,t)\,.$$

Theorem 13.41: Implicit Function Theorem (Special case)

Let F be a function of n variables $(x_1, x_2, \cdots, x_n)$ such that F_{x_1}, F_{x_2}, $\cdots$, F_{x_n} are continuous in a neighborhood of $(a_1, a_2, \cdots, a_n$. If $F(a_1, a_2, \cdots, a_n) = 0$ and $F_{x_n}(a_1, a_2, \cdots, a_n) \neq 0$, then locally near $(a_1, a_2, \cdots, a_n)$ there exists a unique continuous function f satisfying $F(x_1, \cdots, x_{n-1}, f(x_1, \cdots, x_{n-1})) = 0$ and $a_n = f(a_1, \cdots, a_{n-1})$. Moreover, for $1 \leqslant j \leqslant n-1$,

$$\frac{\partial f}{\partial x_j}(x_1, \cdots, x_{n-1}) = -\frac{F_{x_j}(x_1, \cdots, x_{n-1}, f(x_1, \cdots, x_{n-1}))}{F_{x_n}(x_1, \cdots, x_{n-1}, f(x_1, \cdots, x_{n-1}))}\,.$$

Example 13.42. Find $\dfrac{dy}{dx}$ if (x, y) satisfies $y^3 + y^2 - 5y - x^2 + 4 = 0$.

Let $F(x, y) = y^3 + y^2 - 5y - x^2 + 4$. Then $F_x(x, y) = -2x$ and $F_y(x, y) = 3y^2 + 2y - 5$. Therefore,

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)} = \frac{2x}{3y^2 + 2y - 5}\,.$$

Example 13.43. Find $\dfrac{\partial z}{\partial x}$ and $\dfrac{\partial z}{\partial y}$ if (x, y, z) satisfies $3x^2z - x^2y^2 + 2z^3 + 3yz - 5 = 0$.

Let $F(x,y,z) = 3x^2z - x^2y^2 + 2z^3 + 3yz - 5$. Then $F_x(x,y,z) = 6xz - 2xy^2$, $F_y(x,y,z) = -2x^2y + 3z$ and $F_z(x,y,z) = 3x^2 + 6z^2 + 3y$. Therefore,

$$\frac{\partial z}{\partial x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)} = \frac{2xy^2 - 6xz}{3x^2 + 6z^2 + 3y}$$

and

$$\frac{\partial z}{\partial y} = -\frac{F_y(x,y,z)}{F_z(x,y,z)} = \frac{2x^2y - 3z}{3x^2 + 6z^2 + 3y}\,.$$

13.6 Directional Derivatives and Gradients

Let f be a function of two variables. From the discussion above we know that the existence of f_x and f_y does not guarantee the differentiability of f. Since f_x and f_y are the rate of change of the function f in two special directions $(1,0)$ and $(0,1)$, we can ask ourselves whether f is differentiable if the rate of change of f exist in all direction.

Definition 13.44

Let f be a function of two variables x and y, and let $\boldsymbol{u} = \cos\theta\boldsymbol{i} + \sin\theta\boldsymbol{j}$, where $\boldsymbol{i} = (1,0)$ and $\boldsymbol{j} = (0,1)$, be a unit vector. The directional derivative of f in the direction of $\boldsymbol{u}$ at (a,b), denoted by $(D_{\boldsymbol{u}}f)(a,b)$, is the limit

$$(D_{\boldsymbol{u}}f)(a,b) = \lim_{h\to 0} \frac{f(a + h\cos\theta, b + h\sin\theta) - f(a,b)}{h}$$

provided this limit exists.

Example 13.45. Find the direction derivative of $f(x,y) = x^2\sin 2y$ at $\big(1, \frac{\pi}{2}\big)$ in the direction of $\boldsymbol{v} = 3\boldsymbol{i} - 4\boldsymbol{j}$.

We first normalize the vector $\boldsymbol{v}$ and find that $\boldsymbol{u} = \frac{3}{5}\boldsymbol{i} - \frac{4}{5}\boldsymbol{j}$ is in the same direction of $\boldsymbol{v}$ and has unit length. Therefore, for $h \neq 0$,

$$\frac{f\big(1 + \frac{3h}{5}, \frac{\pi}{2} - \frac{4h}{5}\big) - f\big(1, \frac{\pi}{2}\big)}{h} = \frac{(1 + \frac{3h}{5})^2 \sin\big(\pi - \frac{8h}{5}\big) - 1^2\sin\pi}{h} = \Big(1 + \frac{3h}{5}\Big)^2 \frac{\sin\frac{8h}{5}}{h}\,;$$

thus by the fact that $\lim\limits_{h\to 0}\frac{\sin h}{h} = 1$, we find that

$$\lim_{h\to 0} \frac{f\big(1 + \frac{3h}{5}, \frac{\pi}{2} - \frac{4h}{5}\big) - f\big(1, \frac{\pi}{2}\big)}{h} = \lim_{h\to 0}\Big(1 + \frac{3h}{5}\Big)^2 \frac{\sin\frac{8h}{5}}{h} = \frac{8}{5}\,.$$

When f is differentiable, the directional derivative can be computed using the chain rule, and we have the following

Theorem 13.46

Let $R \subseteq \mathbb{R}^2$ be an open region in the plane, and $f : R \to \mathbb{R}$ be a function of two variables. If f is differentiable at $(x_0, y_0) \in R$, then for all unit vector $\boldsymbol{v} = \cos\theta \boldsymbol{i} + \sin\theta \boldsymbol{j}$,

$$(D_{\boldsymbol{u}} f)(x_0, y_0) = f_x(x_0, y_0)\cos\theta + f_y(x_0, y_0)\sin\theta = (Df)(x_0, y_0) \cdot \boldsymbol{u}\,.$$

Proof. Let $g(t) = f(x_0 + t\cos\theta, y_0 + t\sin\theta)$. Then by the chain rule for functions of two variables,

$$(D_{\boldsymbol{u}} f)(x_0, y_0) = \lim_{h\to 0} \frac{g(h) - g(0)}{h} = g'(0) = f_x(x_0, y_0)\cos\theta + f_y(x_0, y_0)\sin\theta\,. \qquad \square$$

Example 13.47. In this example we re-compute of the direction derivative in Example 13.45 using Theorem 13.46. Note that $f(x, y) = x^2 \sin 2y$ is differentiable on $\mathbb{R}^2$ since $f_x(x, y) = 2x\sin 2y$ and $f_y(x, y) = 2x^2 \cos 2y$ are continuous (so that Theorem 13.35 guarantees the differentiability of f). Therefore, Theorem 13.46 implies that

$$(D_{\boldsymbol{u}} f)\big(1, \frac{\pi}{2}\big) = \frac{3}{5} f_x\big(1, \frac{\pi}{2}\big) - \frac{4}{5} f_y\big(1, \frac{\pi}{2}\big) = \frac{3}{5} \cdot 2 \cdot \sin\pi - \frac{4}{5} \cdot 2 \cdot 1^2 \cdot \cos\pi = \frac{8}{5}\,.$$

Unfortunately, the existence of directional derivative of f in all directions does not imply the differentiability of f.

Example 13.48. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be given by

$$f(x, y) = \begin{cases} \dfrac{xy^2}{x^2 + y^4} & \text{if } (x, y) \neq (0, 0)\,, \\ 0 & \text{if } (x, y) = (0, 0)\,, \end{cases}$$

and $\boldsymbol{u} = (\cos\theta, \sin\theta) \in \mathbb{R}^2$ be a unit vector. Then if $\cos\theta \neq 0$ $\big($or equivalently, $\theta \neq \frac{\pi}{2}, \frac{3\pi}{2}\big)$,

$$(D_{\boldsymbol{u}} f)(0, 0) = \lim_{h\to 0} \frac{f(h\cos\theta, h\sin\theta) - f(0, 0)}{h} = \lim_{h\to 0} \frac{h^3 \cos\theta \sin\theta^2}{h(h^2\cos\theta^2 + h^4 \boldsymbol{u}_2^4)} = \frac{\sin\theta^2}{\cos\theta}$$

while if $\cos\theta = 0$,

$$(D_{\boldsymbol{u}} f)(0, 0) = \lim_{h\to 0} \frac{f(h\cos\theta, h\sin\theta) - f(0, 0)}{h} = 0\,.$$

Therefore, the directional derivative of f at $(0,0)$ exist in all directions. However, f is not continuous at $(0,0)$ since

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} f(x,y) = 0$$

and

$$\lim_{\substack{(x,y)\to(0,0)\\ x=y^2}} f(x,y) = \lim_{y\to 0} \frac{y^2 \cdot y^2}{y^4 + y^4} = \frac{1}{2}$$

which shows that the limit of f at $(0,0)$ does not exist.

Definition 13.49

Let $z = f(x,y)$ be a function of x and y such that $f_x(a,b)$ and $f_y(a,b)$ exists. Then the gradient of f at (a,b), denoted by $(\nabla f)(a,b)$ or $(\mathbf{grad} f)(a,b)$, is the vector $(f_x(a,b), f_y(a,b))$; that is,

$$(\nabla f)(a,b) = \big(f_x(a,b), f_y(a,b)\big) = f_x(a,b)\boldsymbol{i} + f_y(a,b)\boldsymbol{j}\,.$$

- **Functions of three variables**

Definition 13.50

Let f be a function of three variables. The directional derivative of f at (a,b,c) in the direction $\boldsymbol{u} = (u_1, u_2, u_3)$, where $u_1^2 + u_2^2 + u_3^2 = 1$, is the limit

$$(D_{\boldsymbol{u}} f)(a,b,c) = \lim_{h\to 0} \frac{f(a+hu_1, b+hu_2, c+hu_3) - f(a,b,c)}{h}$$

provided that the limit exists. The gradient of f at (a,b,c) is $(\nabla f)(a,b,c) = \big(f_x(a,b,c), f_y(a,b,c), f_z(a,b,c)\big)$.

Theorem 13.51

Let f be a function of three variables. If f is differentiable at (a,b,c) and $\boldsymbol{u}$ is a unit vector, then

$$(D_{\boldsymbol{u}} f)(a,b,c) = (\nabla f)(a,b,c) \cdot \boldsymbol{u}\,.$$

13.7 Tangent Planes and Normal Lines

- **The tangent plane of surfaces**

Any three points in the space that are not collinear defines a plane. Suppose that $\mathcal{S}$ is a "surface" (which we have not define yet, but please use the common sense to think about it), and $P_0 = (x_0, y_0, z_0)$ is a point on the plane. Given another two point $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ on the surface such that P_0, P_1, P_2 are not collinear, let $T_{P_1P_2}$ denote the plane determined by P_0, P_1 and P_2. If the plane "approaches" a certain plane as P_1, P_2 approaches P_0, the "limit" is called the tangent plane of $\mathcal{S}$ at P_0.