微積分 MA1002-A 上課筆記（精簡版）

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Definition 13.31

Let $R \subseteq \mathbb{R}^2$ be an open region in the plane, and $f : R \to \mathbb{R}$ be a function of two variables. For $(x_0, y_0) \in R$, f is said to be differentiable at (x_0, y_0) if ($f_x(x_0, y_0)$, $f_y(x_0, y_0)$ both exist and)

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\big|f(x,y) - f(x_0,y_0) - (f_x(x_0,y_0), f_y(x_0,y_0)) \cdot (x - x_0, y - y_0)\big|}{\sqrt{(x-x_0)^2 + (y-y_0)^2}} = 0\,.$$

The ordered pair $\big(f_x(x_0, y_0), f_y(x_0, y_0)\big)$ is called the derivative of f at (x_0, y_0) if f is differentiable at (x_0, y_0) and is usually denoted by $(Df)(x_0, y_0)$.

Let $R \subseteq \mathbb{R}^2$ be an open region in the plane, and $f : R \to \mathbb{R}$ be a function of two variables. For $(x_0, y_0) \in R$, f is said to be differentiable at (x_0, y_0) if ($f_x(x_0, y_0)$, $f_y(x_0, y_0)$ both exist and) there exist functions ε_1 and ε_2 such that

$$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y\,,$$

where both ε_1 and ε_2 approaches 0 as $(x, y) \to (x_0, y_0)$.

- **Differentiability of functions of several variables**

A real-valued function f of n variables is differentiable at $(a_1, a_2, \cdots, a_n)$ if there exist n real numbers $A_1, A_2, \cdots, A_n$ such that

$$\lim_{(x_1,\cdots,x_n)\to(a_1,\cdots,a_n)} \frac{\big|f(x_1,\cdots,x_n) - f(a_1,\cdots,a_n) - (A_1,\cdots,A_n) \cdot (x_1 - a_1, \cdots, x_n - a_n)\big|}{\sqrt{(x_1-a_1)^2 + \cdots + (x_n-a_n)^2}} = 0\,.$$

We also note that when f is differentiable at $(a_1, \cdots, a_n)$, then these numbers $A_1, A_2, \cdots, A_n$ must be $f_{x_1}(a_1, \cdots, a_n)$, $f_{x_2}(a_1, \cdots, a_n)$, $\cdots$, $f_{x_n}(a_1, \cdots, a_n)$, respectively.

Theorem 13.35

Let $R \subseteq \mathbb{R}^2$ be an open region in the plane, and $f : R \to \mathbb{R}$ be a function of two variables. If f_x and f_y are continuous in a neighborhood of $(x_0, y_0) \in R$, then f is differentiable at (x_0, y_0). In particular, if f_x and f_y are continuous on R, then f is differentiable on R; that is, f is said to be differentiable at every point in R.

Theorem 13.36

Let $R \subseteq \mathbb{R}^2$ be an open region in the plane, and $f : R \to \mathbb{R}$ be a function of two variables. If f is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).

13.5 Chain Rules for Functions of Several Variables

Recall the chain rule for functions of one variable:

> Let I, J be open intervals, $f : J \to \mathbb{R}$, $g : I \to \mathbb{R}$ be real-valued functions, and the range of g is contained in J. If g is differentiable at $c \in I$ and f is differentiable at $g(c)$, then $f \circ g$ is differentiable at c and
> $$\frac{d}{dx}\Big|_{x=c}(f \circ g)(x) = f'(g(c))g'(c)\,.$$

For functions of two variables, we have the following

Theorem 13.37

Let $z = f(x, y)$ be a differentiable function (of x and y). If $x = g(t)$ and $y = h(t)$ are differentiable functions (of t), then $z(t) = f\big(x(t), y(t)\big)$ is differentiable and

$$z'(t) = f_x\big(x(t), y(t)\big)x'(t) + f_y\big(x(t), y(t)\big)y'(t)\,.$$

Let $\gamma(t) = \big(x(t), y(t)\big)$. Then $\gamma'(t) = \big(x'(t), y'(t)\big)$, and the chain rule above can be written as

$$\frac{d}{dt}(f \circ \gamma)(t) = (Df)(\gamma(t)) \cdot \gamma'(t)\,.$$

A short-hand notation of the identity above

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = (f_x, f_y) \cdot (x', y')\,.$$

Corollary 13.38

Let $z = f(x, y)$ be a differentiable function (of x and y).

1. If $x = u(s, t)$ and $y = v(s, t)$ are such that $\dfrac{\partial u}{\partial s}$ and $\dfrac{\partial v}{\partial s}$ exist, then the first partial derivative $\dfrac{\partial z}{\partial s}$ of the function $z(s, t) = f\big(u(s, t), v(s, t)\big)$ exists and
$$z_s(s, t) = f_x\big(u(s, t), v(s, t)\big)u_s(s, t) + f_y\big(u(s, t), v(s, t)\big)v_s(s, t)\,.$$

2. If $x = u(s, t)$ and $y = v(s, t)$ are such that $\dfrac{\partial u}{\partial t}$ and $\dfrac{\partial v}{\partial t}$ exist, then the first partial derivative $\dfrac{\partial z}{\partial t}$ of the function $z(s, t) = f\big(u(s, t), v(s, t)\big)$ exists and
$$z_t(s, t) = f_x\big(u(s, t), v(s, t)\big)u_t(s, t) + f_y\big(u(s, t), v(s, t)\big)v_t(s, t)\,.$$

Example 13.39. Let $f(x,y) = x^2y - y^2$. Find $\dfrac{dz}{dt}$, where $z(t) = f(\sin t, e^t)$.

1. Since $z(t) = e^t \sin^2 t - e^{2t}$, by the product rule and the chain rule for functions of one variable, we find that

$$z'(t) = \frac{de^t}{dt}\sin^2 t + e^t \frac{d\sin^2 t}{dt} - 2e^{2t} = e^t \sin^2 t + 2e^t \sin t \cos t - 2e^{2t}\,.$$

2. By the chain rule for functions of two variables,

$$\begin{aligned} z'(t) &= \big(f_x(\sin t, e^t), f_y(\sin t, e^t)\big) \cdot \frac{d}{dt}(\sin t, e^t) \\ &= (2xy, x^2 - 2y)\Big|_{(x,y)=(\sin t, e^t)} \cdot (\cos t, e^t) \\ &= (2e^t \sin t, \sin^2 t - 2e^t) \cdot (\cos t, e^t) \\ &= 2e^t \sin t \cos t + e^t \sin^2 t - 2e^{2t}\,. \end{aligned}$$

Example 13.40. Let $f(x,y) = 2xy$. Find $\dfrac{\partial z}{\partial s}$ and $\dfrac{\partial z}{\partial t}$, where $z(s,t) = f\big(s^2 + t^2, \frac{s}{t}\big)$.

1. Since $z(s,t) = 2(s^2+t^2)\dfrac{s}{t} = \dfrac{2s^3}{t} + 2st$, by the product rule we find that

$$\frac{\partial z}{\partial s}(s,t) = \frac{6s^2}{t} + 2t \qquad \text{and} \qquad \frac{\partial z}{\partial t}(s,t) = -\frac{2s^3}{t^2} + 2s\,.$$

2. By the chain rule for functions of two variables,

$$\begin{aligned} \frac{\partial z}{\partial s}(s,t) &= \big(f_x(s^2+t^2, s/t), f_y(s^2+t^2, s/t)\big) \cdot \frac{\partial}{\partial s}\big(s^2+t^2, \frac{s}{t}\big) \\ &= \big(\frac{2s}{t}, 2(s^2+t^2)\big) \cdot \big(2s, \frac{1}{t}\big) = \frac{4s^2}{t} + \frac{2s^2+2t^2}{t} = \frac{6s^2}{t} + 2t \end{aligned}$$

and

$$\begin{aligned} \frac{\partial z}{\partial t}(s,t) &= \big(f_x(s^2+t^2, s/t), f_y(s^2+t^2, s/t)\big) \cdot \frac{\partial}{\partial t}\big(s^2+t^2, \frac{s}{t}\big) \\ &= \big(\frac{2s}{t}, 2(s^2+t^2)\big) \cdot \big(2t, -\frac{s}{t^2}\big) = 4s - \frac{2s^3+2st^2}{t^2} = -\frac{2s^3}{t^2} + 2s\,. \end{aligned}$$

- **The chain rule for functions of several variables**

Suppose that $w = f(x_1, x_2, \cdots, x_n)$ be a differentiable function (of $x_1, x_2, \cdots, x_n$). If each x_i is a differentiable function of m variables $t_1, t_2, \cdots, t_m$, then

$$\frac{\partial w}{\partial t_1} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_1} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_1} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_1} = \sum_{j=1}^{n} \frac{\partial w}{\partial x_j}\frac{\partial x_j}{\partial t_1},$$

$$\frac{\partial w}{\partial t_2} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_2} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_2} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_2} = \sum_{j=1}^{n} \frac{\partial w}{\partial x_j}\frac{\partial x_j}{\partial t_2},$$

$$\vdots$$

$$\frac{\partial w}{\partial t_m} = \frac{\partial w}{\partial x_1}\frac{\partial x_1}{\partial t_m} + \frac{\partial w}{\partial x_2}\frac{\partial x_2}{\partial t_m} + \cdots + \frac{\partial w}{\partial x_n}\frac{\partial x_n}{\partial t_m} = \sum_{j=1}^{n} \frac{\partial w}{\partial x_j}\frac{\partial x_j}{\partial t_m}.$$

Using the notation of the matrix multiplication,

$$\begin{bmatrix} \frac{\partial w}{\partial t_1} & \frac{\partial w}{\partial t_2} & \cdots & \frac{\partial w}{\partial t_m} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix} \begin{bmatrix} \frac{\partial x_1}{\partial t_1} & \frac{\partial x_1}{\partial t_2} & \cdots & \frac{\partial x_1}{\partial t_m} \\ \frac{\partial x_2}{\partial t_1} & \frac{\partial x_2}{\partial t_2} & \cdots & \frac{\partial x_2}{\partial t_m} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial t_1} & \frac{\partial x_n}{\partial t_2} & \cdots & \frac{\partial x_n}{\partial t_m} \end{bmatrix}.$$

- **Implicit partial differentiation**

In Section 2.4 we have talked about finding derivatives of a function $y = f(x)$ which is defined implicitly by $F(x, y) = 0$ (when F is giving explicitly). Now suppose that $z = F(x, y)$ is a differentiable function and the relation $F(x, y) = 0$ defines a differentiable function $y = f(x)$ implicitly (so that $F(x, f(x)) = 0$). By the chain rule,

$$0 = \frac{d}{dx}F(x, f(x)) = F_x(x, f(x)) + F_y(x, f(x))f'(x)$$

which implies that

$$f'(x) = -\frac{F_x(x, f(x))}{F_y(x, f(x))} \quad \text{if} \quad F_y(x, f(x)) \neq 0.$$

Since f is in general unknown (but exists), we usually write the identity above as

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)} \quad \text{if } F(x, y) = 0 \text{ and } F_y(x, y) \neq 0.$$

In fact, when F_x and F_y are continuous in an open region R, and $F(a,b)=0$ and $F_y(a,b)\neq 0$ at some point $(a,b)\in R$, the relation $F(x,y)=0$ defines a function $y=f(x)$ implicitly near (a,b) and f is continuously differentiable near $x=a$. This is the Implicit Function Theorem and the precise statement is stated as follows.

Theorem 13.41: Implicit Function Theorem (Special case)

Let $R\subseteq\mathbb{R}^2$ be an open region in the plane, and $F:R\to\mathbb{R}$ be a function of two variables such that F_x and F_y are continuous in a neighborhood of $(a,b)\in R$. If $F(a,b)=0$ and $F_y(a,b)=0$, then there exists $\delta>0$ and a unique continuous function $f:(a-\delta,a+\delta)\to\mathbb{R}$ satisfying $F(x,f(x))=0$ for all $x\in(a-\delta,a+\delta)$, and $b=f(a)$. Moreover, f is differentiable on $(a-\delta,a+\delta)$, and

$$f'(x)=-\frac{F_x(x,f(x))}{F_y(x,f(x))}\qquad\forall\, x\in(a-\delta,a+\delta)\,.$$

In general, let F be a function of n variables $(x_1,x_2,\cdots,x_n)$ such that F_{x_1}, F_{x_2}, $\cdots$, F_{x_n} are continuous in a neighborhood of $(a_1,a_2,\cdots,a_n$. If $F(a_1,a_2,\cdots,a_n)=0$ and $F_{x_n}(a_1,a_2,\cdots,a_n)\neq 0$, then locally near $(a_1,a_2,\cdots,a_n)$ there exists a unique continuous function f satisfying $F(x_1,\cdots,x_{n-1},f(x_1,\cdots,x_{n-1}))=0$ and $a_n=f(a_1,\cdots,a_{n-1})$. Moreover, for $1\leqslant j\leqslant n-1$,

$$\frac{\partial f}{\partial x_j}(x_1,\cdots,x_{n-1})=-\frac{F_{x_j}(x_1,\cdots,x_{n-1},f(x_1,\cdots,x_{n-1}))}{F_{x_n}(x_1,\cdots,x_{n-1},f(x_1,\cdots,x_{n-1}))}\,.$$