微積分 MA1002-A 上課筆記（精簡版）

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Definition 13.23

Let f be a function of two variable. The first partial derivative of f with respect to x at (x_0, y_0), denoted by $f_x(x_0, y_0)$, is defined by

$$f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

provided the limit exists. The first partial derivative of f with respect to y at (x_0, y_0), denoted by $f_y(x_0, y_0)$, is defined by

$$f_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

provided the limit exists. When f_x and f_y exist for all (x_0, y_0) (in a certain open region), f_x and f_y are simply called the first partial derivative of f with respect to x and y, respectively.

Theorem 13.28

If f is a function of x and y such that f_{xy} and f_{yx} are continuous on an open disk D, then

$$f_{xy}(x, y) = f_{yx}(x, y) \qquad \forall\, (x, y) \in D\,.$$

Definition 13.30

Let $R \subseteq \mathbb{R}^2$ be an open region in the plane, and $f : R \to \mathbb{R}$ be a function of two variables. For $(x_0, y_0) \in R$, f is said to be differentiable at (x_0, y_0) if there exist real numbers A, B such that

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\big|f(x, y) - f(x_0, y_0) - (A, B)\cdot(x - x_0, y - y_0)\big|}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0\,.$$

If f is differentiable at (x_0, y_0), then $A = f_x(x_0, y_0)$ and $B = f_y(x_0, y_0)$; thus if f is differentiable at (x_0, y_0), $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ both exist and we have the following alternative

Definition 13.31

Let $R \subseteq \mathbb{R}^2$ be an open region in the plane, and $f : R \to \mathbb{R}$ be a function of two variables. For $(x_0, y_0) \in R$, f is said to be differentiable at (x_0, y_0) if ($f_x(x_0, y_0)$, $f_y(x_0, y_0)$ both exist and)

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\big|f(x, y) - f(x_0, y_0) - (f_x(x_0, y_0), f_y(x_0, y_0))\cdot(x - x_0, y - y_0)\big|}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0\,.$$

Remark 13.32. 1. The ordered pair $\big(f_x(x_0,y_0), f_y(x_0,y_0)\big)$ is called the derivative of f at (x_0,y_0) if f is differentiable at (x_0,y_0) and is usually denoted by $(Df)(x_0,y_0)$.

2. Using ε-δ notation, we find that f is differentiable at (x_0,y_0) if for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\begin{aligned}&\big|f(x,y) - f(x_0,y_0) - f_x(x_0,y_0)(x-x_0) - f_y(x_0,y_0)(y-y_0)\big| \\ &\qquad \leqslant \varepsilon\sqrt{(x-x_0)^2+(y-y_0)^2} \qquad \text{whenever} \quad \sqrt{(x-x_0)^2+(y-y_0)^2} < \delta\,.\end{aligned}$$

Now suppose that f is a function of two variables such that $f_x(x_0,y_0)$ and $f_y(x_0,y_0)$ exist. Define

$$\varepsilon(x,y) = \begin{cases} \dfrac{f(x,y) - f(x_0,y_0) - f_x(x_0,y_0)(x-x_0) - f_y(x_0,y_0)(y-y_0)}{\sqrt{(x-x_0)^2+(y-y_0)^2}} & \text{if } (x,y) \neq (x_0,y_0)\,, \\ 0 & \text{if } (x,y) = (x_0,y_0)\,. \end{cases}$$

Let $\Delta x = x - x_0$, $\Delta y = y - y_0$ and $\Delta z = f(x,y) - f(x_0,y_0)$. Then

$$\Delta z = f_x(x_0,y_0)\Delta x + f_y(x_0,y_0)\Delta y + \varepsilon(x,y)\sqrt{\Delta x^2 + \Delta y^2}\,,$$

and f is differentiable at (x_0,y_0) if and only if $\lim\limits_{(x,y)\to(x_0,y_0)} \varepsilon(x,y) = 0$.

Finally, define

$$\varepsilon_1(x,y) = \begin{cases} \dfrac{\varepsilon(x,y)\Delta x}{\sqrt{\Delta x^2+\Delta y^2}} & \text{if } (x,y) \neq (x_0,y_0)\,, \\ 0 & \text{if } (x,y) \neq (x_0,y_0)\,, \end{cases}$$
$$\varepsilon_2(x,y) = \begin{cases} \dfrac{\varepsilon(x,y)\Delta y}{\sqrt{\Delta x^2+\Delta y^2}} & \text{if } (x,y) \neq (x_0,y_0)\,, \\ 0 & \text{if } (x,y) \neq (x_0,y_0)\,, \end{cases} ,$$

then

$$0 \leqslant |\varepsilon_1(x,y)|, |\varepsilon_2(x,y)| \leqslant |\varepsilon(x,y)| = \sqrt{\varepsilon_1(x,y)^2 + \varepsilon_2(x,y)^2}$$

thus the Squeeze Theorem shows that

$$\lim_{(x,y)\to(x_0,y_0)} \varepsilon(x,y) = 0 \quad \text{if and only if} \quad \lim_{(x,y)\to(x_0,y_0)} \varepsilon_1(x,y) = \lim_{(x,y)\to(x_0,y_0)} \varepsilon_2(x,y) = 0\,.$$

By the fact that $\varepsilon(x,y)\sqrt{\Delta x^2+\Delta y^2} = \varepsilon_1(x,y)\Delta x + \varepsilon_2(x,y)\Delta y$, the alternative definition above can be rewritten as

> Let $R \subseteq \mathbb{R}^2$ be an open region in the plane, and $f : R \to \mathbb{R}$ be a function of two variables. For $(x_0, y_0) \in R$, f is said to be differentiable at (x_0, y_0) if ($f_x(x_0, y_0)$, $f_y(x_0, y_0)$ both exist and) there exist functions ε_1 and ε_2 such that
>
> $$\Delta z = f_x(x_0, y_0)\Delta x + f_y(x_0, y_0)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y\,,$$
>
> where both ε_1 and ε_2 approaches 0 as $(x, y) \to (x_0, y_0)$.

Example 13.33. Show that the function $f(x, y) = x^2 + 3y$ is differentiable at every point in the plane.

Let $(a, b) \in \mathbb{R}^2$ be given. Then $f_x(a, b) = 2a$ and $f_y(a, b) = 3$. Therefore,

$$\begin{aligned}\Delta z - f_x(a, b)\Delta x - f_y(a, b)\Delta y &= x^2 + 3y - a^2 - 3b - 2a(x - a) - 3(y - b)\\ &= (x - a)^2 = \varepsilon_1(x, y)\Delta x + \varepsilon_2(x, y)\Delta y\,,\end{aligned}$$

where $\varepsilon_1(x, y) = x - a$ and $\varepsilon_2(x, y) = 0$. Since

$$\lim_{(x,y)\to(a,b)} \varepsilon_1(x, y) = 0 \qquad \text{and} \qquad \lim_{(x,y)\to(a,b)} \varepsilon_2(x, y) = 0\,,$$

by the definition we find that f is differentiable at (a, b).

Example 13.34. The function f given in Example 13.25 is differentiable at $(0, 0)$ since if $(x, y) \neq (0, 0)$,

$$\frac{\big|f(x, y) - f(0, 0) - f_x(0, 0)x - f_y(0, 0)y\big|}{\sqrt{x^2 + y^2}} = \frac{\big|xy(x^2 - y^2)\big|}{(x^2 + y^2)^{\frac{3}{2}}} \leqslant \frac{|x^2 - y^2|}{\sqrt{x^2 + y^2}} \leqslant |x| + |y|$$

and the Squeeze Theorem shows that

$$\lim_{(x,y)\to(0,0)} \frac{\big|f(x, y) - f(0, 0) - f_x(0, 0)(x - 0) - f_y(0, 0)(y - 0)\big|}{\sqrt{x^2 + y^2}} = 0\,.$$

• Differentiability of functions of several variables

A real-valued function f of n variables is differentiable at $(a_1, a_2, \cdots, a_n)$ if there exist n real numbers $A_1, A_2, \cdots, A_n$ such that

$$\lim_{(x_1,\cdots,x_n)\to(a_1,\cdots,a_n)} \frac{\big|f(x_1, \cdots, x_n) - f(a_1, \cdots, a_n) - (A_1, \cdots, A_n) \cdot (x_1 - a_1, \cdots, x_n - a_n)\big|}{\sqrt{(x_1 - a_1)^2 + \cdots + (x_n - a_n)^2}} = 0\,.$$

We also note that when f is differentiable at $(a_1, \cdots, a_n)$, then these numbers $A_1, A_2, \cdots, A_n$ must be $f_{x_1}(a_1, \cdots, a_n)$, $f_{x_2}(a_1, \cdots, a_n)$, $\cdots$, $f_{x_n}(a_1, \cdots, a_n)$, respectively.

It is usually easier to compute the partial derivatives of a function of several variables than determine the differentiability of that function. Is there any connection between some specific properties of partial derivatives and the differentiability? We have the following

Theorem 13.35

Let $R \subseteq \mathbb{R}^2$ be an open region in the plane, and $f : R \to \mathbb{R}$ be a function of two variables. If f_x and f_y are continuous in a neighborhood of $(x_0, y_0) \in R$, then f is differentiable at (x_0, y_0). In particular, if f_x and f_y are continuous on R, then f is differentiable on R; that is, f is said to be differentiable at every point in R.

Therefore, the differentiability of f in Example 13.25 at any point $(x_0, y_0) \neq (0, 0)$ can be guaranteed since f_x and f_y are continuous on $\mathbb{R}^2 \backslash \{(0, 0)\}$.

Theorem 13.36

Let $R \subseteq \mathbb{R}^2$ be an open region in the plane, and $f : R \to \mathbb{R}$ be a function of two variables. If f is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).

Proof. By the definition of differentiability, if f is differentiable at (x_0, y_0), then there exists function ε_1 and ε_2 such that

$$\lim_{(x,y)\to(x_0,y_0)} \varepsilon_1(x, y) = \lim_{(x,y)\to(x_0,y_0)} \varepsilon_w(x, y) = 0$$

and

$$\begin{aligned} f(x, y) = f(x_0, y_0) &+ f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) \\ &+ \varepsilon_1(x, y)(x - x_0) + \varepsilon_2(x, y)(y - y_0) \,. \end{aligned}$$

Then $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = f(x_0, y_0)$. □

Example 13.36. Consider the function

$$f(x, y) = \begin{cases} \dfrac{-3xy}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0) \,, \\ 0 & \text{if } (x, y) = (0, 0) \,. \end{cases}$$

Then f is not continuous at $(0,0)$ since

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} f(x,y) = 0 \qquad \text{but} \qquad \lim_{\substack{(x,y)\to(0,0)\\ x=y}} f(x,y) = -\frac{3}{2}\,.$$

However, we note that

$$f_x(0,0) = \lim_{\Delta x\to 0} \frac{f(\Delta x,0)-f(0,0)}{\Delta x} = 0 \quad \text{and} \quad f_y(0,0) = \lim_{\Delta y\to 0} \frac{f(0,\Delta y)-f(0,0)}{\Delta y} = 0\,.$$

Therefore, the existence of partial derivatives at a point in all directions does **not** even imply the continuity.