微積分 MA1002-A 上課筆記（精簡版）

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Definition 13.23

Let f be a function of two variable. The first partial derivative of f with respect to x at (x_0, y_0), denoted by $f_x(x_0, y_0)$, is defined by

$$f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

provided the limit exists. The first partial derivative of f with respect to y at (x_0, y_0), denoted by $f_y(x_0, y_0)$, is defined by

$$f_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

provided the limit exists. When f_x and f_y exist for all (x_0, y_0) (in a certain open region), f_x and f_y are simply called the first partial derivative of f with respect to x and y, respectively.

- **Notation**: For $z = f(x, y)$, the partial derivative f_x and f_y, can also be denoted by

$$\frac{\partial}{\partial x} f(x, y) = f_x(x, y) = z_x = \frac{\partial z}{\partial x} = \frac{\partial f}{\partial x}(x, y)\,,$$
$$\frac{\partial}{\partial y} f(x, y) = f_y(x, y) = z_y = \frac{\partial z}{\partial y} = \frac{\partial f}{\partial y}(x, y)\,.$$

When evaluating the partial derivative at (x_0, y_0), we write

$$f_x(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial}{\partial x}\Big|_{(x,y)=(x_0,y_0)} f(x, y)\,,$$
$$f_y(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = \frac{\partial}{\partial y}\Big|_{(x,y)=(x_0,y_0)} f(x, y)\,.$$

Example 13.25. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x, y) = \begin{cases} \dfrac{xy(x^2 - y^2)}{x^2 + y^2} & \text{if } (x, y) \neq (0, 0)\,, \\ 0 & \text{if } (x, y) = (0, 0)\,. \end{cases}$$

Then

$$f_x(x, y) = \begin{cases} \dfrac{x^4 y + 4x^2 y^3 - y^5}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0)\,, \\ 0 & \text{if } (x, y) = (0, 0)\,, \end{cases}$$

and

$$f_y(x, y) = \begin{cases} \dfrac{x^5 - 4x^3 y^2 - xy^4}{(x^2 + y^2)^2} & \text{if } (x, y) \neq (0, 0)\,, \\ 0 & \text{if } (x, y) = (0, 0)\,. \end{cases}$$

- **Higher-order partial derivatives:**

We can also take higher-order partial derivatives of functions of several variables. For example, for $z = f(x, y)$, we have

$$\begin{aligned} f_{xx} &= (f_x)_x = \frac{\partial}{\partial x}\Big(\frac{\partial f}{\partial x}\Big) = \frac{\partial^2 f}{\partial x^2}\,, \\ f_{yy} &= (f_y)_y = \frac{\partial}{\partial y}\Big(\frac{\partial f}{\partial y}\Big) = \frac{\partial^2 f}{\partial y^2}\,, \\ f_{xy} &= (f_x)_y = \frac{\partial}{\partial y}\Big(\frac{\partial f}{\partial x}\Big) = \frac{\partial^2 f}{\partial y \partial x}\,, \\ f_{yx} &= (f_y)_x = \frac{\partial}{\partial x}\Big(\frac{\partial f}{\partial y}\Big) = \frac{\partial^2 f}{\partial x \partial y}\,. \end{aligned}$$

The third and fourth cases are called ***mixed partial derivatives***.

Example 13.26. In this example we compute the second partial derivatives of the function given in 13.25. We have obtained that

$$f_x(x, y) = \begin{cases} \dfrac{x^4 y + 4x^2y^3 - y^5}{(x^2+y^2)^2} & \text{if } (x, y) \neq (0, 0)\,, \\ 0 & \text{if } (x, y) = (0, 0)\,, \end{cases}$$

and

$$f_y(x, y) = \begin{cases} \dfrac{x^5 - 4x^3y^2 - xy^4}{(x^2+y^2)^2} & \text{if } (x, y) \neq (0, 0)\,, \\ 0 & \text{if } (x, y) = (0, 0)\,. \end{cases}$$

If $(x, y) \neq (0, 0)$, the quotient rule, the product rule and the chain rule (for functions of one variable) together show that

$$\begin{aligned} f_{xx}(x, y) &= \frac{(x^2+y^2)^2 \frac{\partial}{\partial x}(x^4y + 4x^2y^3 - y^5) - (x^4y + 4x^2y^3 - y^5)\frac{\partial}{\partial x}(x^2+y^2)^2}{(x^2+y^2)^4} \\ &= \frac{(x^2+y^2)^2(4x^3y + 8xy^3) - (x^4y + 4x^2y^3 - y^5)\cdot\big[2(x^2+y^2)\cdot(2x)\big]}{(x^2+y^2)^3} \\ &= \frac{(x^2+y^2)(4x^3y + 8xy^3) - 4x(x^4y + 4x^2y^3 - y^5)}{(x^2+y^2)^3} = \frac{-4x^3y^3 + 12xy^5}{(x^2+y^2)^3}\,. \end{aligned}$$

Similarly, if $(x,y) \neq (0,0)$,

$$\begin{aligned}
f_{yy}(x,y) &= \frac{(x^2+y^2)^2(-8x^3y-4xy^3)-(x^5-4x^3y^2-xy^4)\cdot\big[2(x^2+y^2)\cdot(2y)\big]}{(x^2+y^2)^2} \\
&= \frac{-12x^5y+4x^3y^3}{(x^2+y^2)^3}\,, \\
f_{xy}(x,y) &= \frac{(x^2+y^2)(x^4+12x^2y^2-5y^4)-4y(x^4y+4x^2y^3-y^5)}{(x^2+y^2)^3} \\
&= \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3} \\
f_{yx}(x,y) &= \frac{(x^2+y^2)(5x^4-12x^2y^2-y^4)-4x(x^5-4x^3y^2-xy^4)}{(x^2+y^2)^3} \\
&= \frac{x^6+9x^4y^2-9x^2y^4-y^6}{(x^2+y^2)^3}\,.
\end{aligned}$$

We note that when $(x,y) \neq (0,0)$, $f_{xy}(x,y) = f_{yx}(x,y)$.

Since $f_x(x,0) = f_y(0,y) = 0$ for all $x \neq 0$, we find that

$$f_{xx}(0,0) = \lim_{\Delta x\to 0} \frac{f_x(\Delta x,0)-f_x(0,0)}{\Delta x} = 0$$

and

$$f_{yy}(0,0) = \lim_{\Delta y\to 0} \frac{f_y(0,\Delta y)-f_y(0,0)}{\Delta y} = 0\,.$$

Finally, we compute $f_{xy}(0,0)$ and $f_{yx}(0,0)$. By definition,

$$f_{xy}(0,0) = \lim_{\Delta y\to 0} \frac{f_x(0,\Delta y)-f_x(0,0)}{\Delta y} = \lim_{\Delta y\to 0} \frac{\frac{-\Delta y^5}{\Delta y^4}}{\Delta y} = -1$$

and

$$f_{yx}(0,0) = \lim_{\Delta x\to 0} \frac{f_y(\Delta x,0)-f_y(0,0)}{\Delta x} = \lim_{\Delta y\to 0} \frac{\frac{\Delta x^5}{\Delta x^4}}{\Delta x} = 1\,.$$

We note that $f_{xy}(0,0) \neq f_{yx}(0,0)$.

Remark 13.27. In the previous example, the mixed derivative $f_{yx}(x,y)$ can be computed using a different method for $(x,y) \neq (0,0)$ (as long as f_{xy} is known). We first note that if $(x,y) \neq (0,0)$, then $f(x,y) = -f(y,x)$; thus

$$\begin{aligned}
f_y(x,y) &= \lim_{h\to 0} \frac{f(x,y+h)-f(x,y)}{h} = \lim_{h\to 0} \frac{-f(y+h,x)+f(y,x)}{h} \\
&= -\lim_{h\to 0} \frac{f(y+h,x)-f(y,x)}{h} = -f_x(y,x)\,.
\end{aligned}$$

Therefore,

$$\begin{aligned} f_{yx}(x,y) &= \lim_{h\to 0} \frac{f_y(x+h,y) - f_y(x,y)}{h} = \lim_{h\to 0} \frac{-f_x(y,x+h) + f_x(y,x)}{h} \\ &= -\lim_{h\to 0} \frac{f_x(y,x+h) - f_x(y,x)}{h} = -f_{xy}(y,x)\,. \end{aligned}$$

Since $f_{xy}(x,y) = \dfrac{x^6 + 9x^4y^2 - 9x^2y^4 - y^6}{(x^2+y^2)^3}$, we have

$$f_{yx}(x,y) = -f_{xy}(y,x) = -\frac{y^6 + 9y^4x^2 - 9y^2x^4 - x^6}{(y^2+x^2)^3} = f_{xy}(x,y) \quad \text{if} \quad (x,y) \neq (0,0).$$

In the previous example, we see that the mixed derivatives f_{xy} and f_{yx} are identical in one case but are not identical in another cases. In general, we have the following

Theorem 13.28

If f is a function of x and y such that f_{xy} and f_{yx} are continuous on an open disk D, then

$$f_{xy}(x,y) = f_{yx}(x,y) \qquad \forall\,(x,y) \in D\,.$$

The theorem above also applies to functions of three or more variables as long as similar condition holds. For example, if $w = f(x,y,z)$ and f_{xy}, f_{yx} are continuous in an "open disk" D on the plane $z = z_0$, then

$$f_{xy}(x,y,z_0) = f_{yx}(x,y,z_0) \qquad \forall\,(x,y) \in D\,.$$

Example 13.29. Let $f(x,y,z) = ye^x + x\ln z$. Then $f_x(x,y,z) = ye^x + \ln z$, $f_y(x,y,z) = e^x$ and $f_z(x,y,z) = \dfrac{x}{z}$. Therefore,

$$\begin{aligned} f_{xy}(x,y,z) &= e^x = f_{yx}(x,y,z)\,, \\ f_{xz}(x,y,z) &= \frac{1}{z} = f_{zx}(x,y,z) \quad \forall\, z \neq 0\,, \\ f_{yz}(x,y,z) &= 0 = f_{zy}(x,y,z)\,. \end{aligned}$$

13.4 Differentiability of Functions of Several Variables

Recall that a function $f : (a,b) \to \mathbb{R}$ is said to be differentiable at a point $c \in (a,b)$ if the limit

$$\lim_{\Delta x\to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

exists. The differentiability of f at c can be rephrased as follows:

> A function $f : (a, b) \to \mathbb{R}$ is said to be differentiable at $c \in (a, b)$ if there exists $m \in \mathbb{R}$ such that
> $$\lim_{\Delta x \to 0} \left| \frac{f(c + \Delta x) - f(c) - m\Delta x}{\Delta x} \right| = 0\,.$$
> or equivalently,
> $$\lim_{x \to c} \left| \frac{f(x) - f(c) - m(x - c)}{x - c} \right| = 0\,.$$

This equivalent way of defining differentiability of functions of one variable motivate the following

Definition 13.30

Let $R \subseteq \mathbb{R}^2$ be an open region in the plane, and $f : R \to \mathbb{R}$ be a function of two variables. For $(x_0, y_0) \in R$, f is said to be differentiable at (x_0, y_0) if there exist real numbers A, B such that

$$\lim_{(x,y) \to (x_0,y_0)} \frac{\big|f(x, y) - f(x_0, y_0) - A(x - x_0) - B(y - y_0)\big|}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} = 0\,.$$

Suppose that f is differentiable at (x_0, y_0). When (x, y) approaches (x_0, y_0) along the line $y = y_0$, we find that

$$\begin{aligned} 0 &= \lim_{\substack{(x,y) \to (x_0,y_0) \\ y = y_0}} \frac{\big|f(x, y) - f(x_0, y_0) - A(x - x_0) - B(y - y_0)\big|}{\sqrt{(x - x_0)^2 + (y - y_0)^2}} \\ &= \lim_{x \to x_0} \frac{\big|f(x, y_0) - f(x_0, y_0) - A(x - x_0)\big|}{|x - x_0|} = \lim_{x \to x_0} \left| \frac{f(x, y_0) - f(x_0, y_0)}{x - x_0} - A \right| \end{aligned}$$

which implies that the number A must be $f_x(x_0, y_0)$. Similarly, $B = f_y(x_0, y_0)$.