微積分 MA1002-A 上課筆記（精簡版）

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Definition 13.10

Let $\delta > 0$ be given. The δ-neighborhood about a point (x_0, y_0) in the plane is the open disk centered at (x_0, y_0) with radius δ given by

$$D\big((x_0, y_0), \delta\big) \equiv \Big\{(x, y) \,\Big|\, \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta\Big\} .$$

Definition 13.11

Let R be a collection of points in the plane. A point (x_0, y_0) (in R) is called an ***interior point*** of R if there exists $\delta > 0$ such that

$$D\big((x_0, y_0), \delta\big) \subseteq R .$$

If every point in R is an interior point of R, then R is called an open region. A point (x_0, y_0) is called a ***boundary point*** of R if every δ-neighborhood about (x_0, y_0) containing points inside R and point outsides R. In other words, (x_0, y_0) is a boundary point of R if

$$\forall\, \delta > 0, D\big((x_0, y_0), \delta\big) \cap R \neq \varnothing \quad \text{and} \quad D\big((x_0, y_0), \delta\big) \cap R^{\complement} \neq \varnothing .$$

If R contains all its boundary points, then R is called a closed region.

Definition 13.13

Let f be a real-valued function of two variables defined, except possibly at (x_0, y_0), on an open disk centered at (x_0, y_0), and let L be a real number. Then

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\big|f(x) - L\big| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta .$$

13.2.1 Continuity of functions of two variables

Definition 13.19

A function f of two variables is ***continuous at a point*** (x_0, y_0) in an open region R if $f(x_0, y_0)$ is defined and is equal to the limit of $f(x, y)$ as (x, y) approaches (x_0, y_0); that is,

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = f(x_0,y_0)\,.$$

In other words, f is continuous at (x_0, y_0) if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left|f(x,y) - f(x_0,y_0)\right| < \varepsilon \quad \text{whenever} \quad \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta\,.$$

The function f is ***continuous in the open region*** R if it is continuous at every point in R.

Remark 13.20. 1. Unlike the case that f does not have to be defined at (x_0, y_0) in order to consider the limit of f at (x_0, y_0), for f to be continuous at a point (x_0, y_0) f has to be defined at (x_0, y_0).

2. A point (x_0, y_0) is called a discontinuity of f if f is not continuous at (x_0, y_0). (x_0, y_0) is called a ***removable discontinuity*** of f if $\lim\limits_{(x,y)\to(x_0,y_0)} f(x,y)$ exists.

Theorem 13.21

Let f and g be functions of two variables such that f and g are continuous at (x_0, y_0).

1. $f \pm g$ is continuous at (x_0, y_0).
2. fg is continuous at (x_0, y_0).
3. $\dfrac{f}{g}$ is continuous at (x_0, y_0) if $g(x_0, y_0) \neq 0$.

Theorem 13.22

If h is continuous at (x_0, y_0) and g is continuous at $h(x_0, y_0)$, then the composite function $g \circ h$ is continuous at (x_0, y_0); that is,

$$\lim_{(x,y)\to(x_0,y_0)} (g \circ h)(x,y) = g\big(h(x_0,y_0)\big)\,.$$

13.3 Partial Derivatives

Definition 13.23

Let f be a function of two variable. The first partial derivative of f with respect to x at (x_0, y_0), denoted by $f_x(x_0, y_0)$, is defined by

$$f_x(x_0, y_0) = \lim_{\Delta x \to 0} \frac{f(x_0 + \Delta x, y_0) - f(x_0, y_0)}{\Delta x}$$

provided the limit exists. The first partial derivative of f with respect to y at (x_0, y_0), denoted by $f_y(x_0, y_0)$, is defined by

$$f_y(x_0, y_0) = \lim_{\Delta y \to 0} \frac{f(x_0, y_0 + \Delta y) - f(x_0, y_0)}{\Delta y}$$

provided the limit exists. When f_x and f_y exist for all (x_0, y_0) (in a certain open region), f_x and f_y are simply called the first partial derivative of f with respect to x and y, respectively.

- **Notation**: For $z = f(x, y)$, the partial derivative f_x and f_y, can also be denoted by

$$\frac{\partial}{\partial x} f(x, y) = f_x(x, y) = z_x = \frac{\partial z}{\partial x} = \frac{\partial f}{\partial x}(x, y)$$

and

$$\frac{\partial}{\partial y} f(x, y) = f_y(x, y) = z_y = \frac{\partial z}{\partial y} = \frac{\partial f}{\partial y}(x, y)\,.$$

When evaluating the partial derivative at (x_0, y_0), we write

$$f_x(x_0, y_0) = \frac{\partial f}{\partial x}(x_0, y_0) = \frac{\partial}{\partial x}\Big|_{(x,y)=(x_0,y_0)} f(x, y)$$

and

$$f_y(x_0, y_0) = \frac{\partial f}{\partial y}(x_0, y_0) = \frac{\partial}{\partial y}\Big|_{(x,y)=(x_0,y_0)} f(x, y)\,.$$

Example 13.24. For $f(x, y) = xe^{x^2 y}$, find f_x and f_y.

Note that f_x is obtained by treating y as a constant and differentiate f with respect to x. Therefore, the product rule implies tat

$$f_x(x, y) = \Big(\frac{\partial}{\partial x} x\Big) e^{x^2 y} + x \Big(\frac{\partial}{\partial x} e^{x^2 y}\Big) = e^{x^2 y} + x \cdot e^{x^2 y} \cdot 2xy = (1 + 2x^2 y) e^{x^2 y}\,.$$

Similarly,

$$f_y(x, y) = \Big(\frac{\partial}{\partial y} x\Big) e^{x^2 y} + x \Big(\frac{\partial}{\partial y} e^{x^2 y}\Big) = x^3 e^{x^2 y}\,.$$

Example 13.25. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{xy(x^2-y^2)}{x^2+y^2} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

Then if $(x,y) \neq (0,0)$, we can apply the quotient rule (and product rule) to compute the partial derivatives and obtain that

$$\begin{aligned} f_x(x,y) &= \frac{(x^2+y^2)\dfrac{\partial}{\partial x}\big[xy(x^2-y^2)\big] - xy(x^2-y^2)\dfrac{\partial}{\partial x}(x^2+y^2)}{(x^2+y^2)^2} \\ &= \frac{(x^2+y^2)\big[y(x^2-y^2)+2x^2y\big] - xy(x^2-y^2)\cdot(2x)}{(x^2+y^2)^2} \\ &= \frac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2}\,. \end{aligned}$$

If $(x,y) = (0,0)$, we cannot use the quotient rule to compute the derivative since the denominate is 0 (so that 4 of Theorem 13.15 cannot be applied), and we have to compute $f_x(0,0)$ using the definition. By definition,

$$f_x(0,0) = \lim_{\Delta x \to 0} \frac{f(\Delta x, 0) - f(0,0)}{\Delta x} = 0\,.$$

Therefore,

$$f_x(x,y) = \begin{cases} \dfrac{x^4y+4x^2y^3-y^5}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

Similarly,

$$f_y(x,y) = \begin{cases} \dfrac{x^5-4x^3y^2-xy^4}{(x^2+y^2)^2} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

• **Geometric meaning of partial derivatives**: Let $f(x,y)$ be a function of two variable, (x_0,y_0) be given, and $z_0 = f(x_0,y_0)$. Consider the graph of the function $z = f(x,y_0)$ (of one variable) on the xz-plane. If the graph $z = f(x,y_0)$ has a tangent line at (x_0,z_0), then the slope of the tangent line at (x_0,z_0) is given by

$$\lim_{h \to 0} \frac{f(x_0+h,y_0) - f(x_0,y_0)}{h}$$

and this limit, if exists, is $f_x(x_0,y_0)$. This is called ***the slopes in the x-direction of the surface $z = f(x,y)$ at the point*** (x_0,y_0,z_0). Similarly, the slope of the tangent line of the graph of $z = f(x_0,y)$ at (y_0,z_0) is $f_y(x_0,y_0)$, and is called ***the slopes in the y-direction of the surface $z = f(x,y)$ at the point*** (x_0,y_0,z_0).

• **Partial derivatives of functions of three or more variables:**

The concept of partial derivatives can be extended to functions of three or more variables. For example, if $w = f(x, y, z)$, then

$$\frac{\partial w}{\partial x} = f_x(x,y,z) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y, z) - f(x,y,z)}{\Delta x}\,,$$
$$\frac{\partial w}{\partial y} = f_y(x,y,z) = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y, z) - f(x,y,z)}{\Delta y}\,,$$
$$\frac{\partial w}{\partial z} = f_z(x,y,z) = \lim_{\Delta z \to 0} \frac{f(x, y, z+\Delta z) - f(x,y,z)}{\Delta z}\,.$$

In general, if $w = f(x_1, x_2, \cdots, x_n)$, then there are n first partial derivatives denoted by

$$\frac{\partial w}{\partial x_k} = f_{x_k}(x_1, x_2, \cdots, x_n)\,, \quad k = 1, 2, \cdots, n\,.$$

• **Higher-order partial derivatives:**

We can also take higher-order partial derivatives of functions of several variables. For example, for $z = f(x, y)$,

1. Differentiate twice with respect to x:
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = f_{xx}\,.$$
2. Differentiate twice with respect to y:
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = f_{yy}\,.$$
3. Differentiate first with respect to x and then with respect to y:
$$\frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = f_{xy}\,.$$
4. Differentiate first with respect to y and then with respect to x:
$$\frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = f_{yx}\,.$$

The third and fourth cases are called ***mixed partial derivatives***.