微積分 MA1002-A 上課筆記（精簡版）

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Definition 13.6: Level Curves

Let D be a set of ordered pairs of real numbers, and $f : D \to \mathbb{R}$ be a function of two variables. A level curve (or contour curve) of f is a collection of points (x, y) in D along which the value of $f(x, y)$ is constant.

Definition 13.7: Level Surfaces

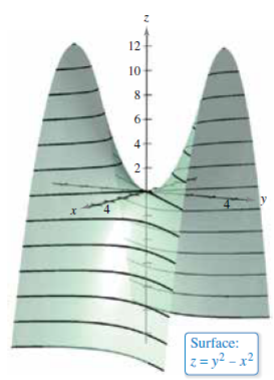
Let D be a set of ordered pairs of real numbers, and $f : D \to \mathbb{R}$ be a function of three variables. A level surface of f is a collection of points (x, y, z) in D along which the value of $f(x, y, z)$ is constant.

Example 13.8. Consider the function $f(x, y, z) = x^2 + y^2 + z^2$. Then the level surface

$$\big\{(x, y, z) \,\big|\, f(x, y, z) = 1\big\}$$

is the unit sphere centered at the origin.

Example 13.9. The graph of $f(x, y) = y^2 - x^2$ is called a hyperbolic paraboloid. A level curve of a hyperbolic paraboloid is a hyperbola (or degenerated hyperbola), and each plane perpendicular to the xy-plane intersects the graph of $z = f(x, y)$ along a parabola (or degenerated parabola).



Hyperbolic paraboloid

13.2 Limits and Continuity

Definition 13.10

Let $\delta > 0$ be given. The δ-neighborhood about a point (x_0, y_0) in the plane is the open disk centered at (x_0, y_0) with radius δ given by

$$D\big((x_0, y_0), \delta\big) \equiv \Big\{(x, y) \,\Big|\, \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta\Big\}.$$

Definition 13.11

Let R be a collection of points in the plane. A point (x_0, y_0) (in R) is called an ***interior point*** of R if there exists $\delta > 0$ such that

$$D\big((x_0, y_0), \delta\big) \subseteq R\,.$$

If every point in R is an interior point of R, then R is called an open region. A point (x_0, y_0) is called a ***boundary point*** of R if every δ-neighborhood about (x_0, y_0) containing points inside R and point outsides R. In other words, (x_0, y_0) is a boundary point of R if

$$\forall\, \delta > 0, D\big((x_0, y_0), \delta\big) \cap R \neq \varnothing \quad \text{and} \quad D\big((x_0, y_0), \delta\big) \cap R^{\complement} \neq \varnothing\,.$$

If R contains all its boundary points, then R is called a closed region.

Remark 13.12. For $x \in \mathbb{R}$ and $\delta > 0$, let $D(x, \delta)$ denote the interval $(x - \delta, x + \delta)$ (and called the interval centered at x with radius r). Then for each $x \in (a, b)$, there exists $\delta > 0$ such that $D(x, r) \subseteq (a, b)$; thus (a, b) is called an open interval. The end-points a, b of the interval are boundary points of the interval, and $[a, b]$ is a closed interval since it contains all its boundary points.

Definition 13.13

Let f be a real-valued function of two variables defined, except possibly at (x_0, y_0), on an open disk centered at (x_0, y_0), and let L be a real number. Then

$$\lim_{(x,y)\to(x_0,y_0)} f(x, y) = L$$

if for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\big|f(x) - L\big| < \varepsilon \quad \text{whenever} \quad 0 < \sqrt{(x - x_0)^2 + (y - y_0)^2} < \delta\,.$$

Remark 13.14. If $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = L_1$ and $\lim\limits_{(x,y)\to(x_0,y_0)} f(x, y) = L_2$, then $L_1 = L_2$. In other words, the limit is unique when it exists.

The proof of the following is almost identical to the one of Theorem 1.13.

Theorem 13.15: Properties of Limits of Functions of Two Variables

Let $(x_0, y_0) \in \mathbb{R}^2$. Suppose that the limits

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L \qquad \text{and} \qquad \lim_{(x,y)\to(x_0,y_0)} g(x,y) = K\,.$$

both exist, and c is a constant.

1. $\displaystyle\lim_{(x,y)\to(x_0,y_0)} c = c$, $\displaystyle\lim_{(x,y)\to(x_0,y_0)} x = x_0$ and $\displaystyle\lim_{(x,y)\to(x_0,y_0)} y = y_0$.
2. $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \big[f(x,y) \pm g(x,y)\big] = L + K$;
3. $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \big[f(x,y)g(x,y)\big] = LK$;
4. $\displaystyle\lim_{(x,y)\to(x_0,y_0)} \frac{f(x,y}{g(x,y)} = \frac{L}{K}$ if $K \neq 0$.

Theorem 13.16: Squeeze

Let $(x_0, y_0) \in \mathbb{R}^2$. Suppose that f, g, h are functions of two variables such that

$$g(x,y) \leqslant f(x,y) \leqslant h(x,y)$$

except possible at (x_0, y_0), and $\displaystyle\lim_{(x,y)\to(x_0,y_0)} g(x,y) = \lim_{(x,y)\to(x_0,y_0)} h(x,y) = L$, then

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L\,.$$

Example 13.17. For $(a,b) \in \mathbb{R}^2$, find the limit $\displaystyle\lim_{(x,y)\to(a,b)} \frac{5x^2y}{x^2+y^2}$.

First we note that 1-3 of Theorem 13.15 implies that the function $f(x,y) = 5x^2y$ and $g(x,y) = x^2 + y^2$ has the property that

$$\lim_{(x,y)\to(a,b)} f(x,y) = 5a^2b \quad \text{and} \quad \lim_{(x,y)\to(a,b)} g(x,y) = a^2 + b^2\,.$$

Therefore, Theorem 13.15 again shows the following:

1. If $(a,b) \neq (0,0)$, then 4 of Theorem 13.15 implies that

$$\lim_{(x,y)\to(a,b)} \frac{5x^2y}{x^2+y^2} = \lim_{(x,y)\to(a,b)} \frac{f(x,y)}{g(x,y)} = \frac{5a^2b}{a^2+b^2}\,.$$

2. If $(a,b) = (0,0)$, then we cannot apply 4 of Theorem 13.15 to compute the limit. Nevertheless, since

$$\left|\frac{5x^2y}{x^2+y^2} - 0\right| \leqslant 5|y| \qquad \forall\,(x,y) \neq (0,0)\,,$$

the Squeeze Theorem implies that

$$\lim_{(x,y)\to(0,0)} \frac{5x^2y}{x^2+y^2} = 0\,.$$

Example 13.18. Show that the limit $\displaystyle\lim_{(x,y)\to(0,0)} \Big(\frac{x^2-y^2}{x^2+y^2}\Big)^2$ does not exist.

Let $f(x,y) = \Big(\dfrac{x^2-y^2}{x^2+y^2}\Big)^2$. By the definition of limits, if $\displaystyle\lim_{(x,y)\to(0,0)} f(x,y) = L$ exists, then there exists $\delta > 0$ such that

$$\big|f(x,y) - L\big| < \frac{1}{2} \quad \text{whenever} \;\; 0 < \sqrt{x^2+y^2} < \delta$$

which implies that

$$L - \frac{1}{2} < f(x,y) < L + \frac{1}{2} \quad \text{whenever} \;\; 0 < \sqrt{x^2+y^2} < \delta\,. \tag{13.2.1}$$

However, when (x,y) satisfies $0 < \sqrt{x^2+y^2} < \delta$ and $x = y$, then $f(x,y) = 0$ while on the other hand, when (x,y) satisfies $0 < \sqrt{x^2+y^2} < \delta$ and $y = 0$, then $f(x,y) = 1$. This is a contradiction because of (13.2.1).

• Another way of looking at this limit: When (x,y) approaches $(0,0)$ along the line $x = y$ (we use the notation $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ x=y}}$ to denote this limit process), we find that

$$\lim_{\substack{(x,y)\to(0,0)\\ x=y}} f(x,y) = 0$$

and when (x,y) approaches $(0,0)$ along the x-axis (we use the notation $\displaystyle\lim_{\substack{(x,y)\to(0,0)\\ y=0}}$ to denote this limit process), we find that

$$\lim_{\substack{(x,y)\to(0,0)\\ y=0}} f(x,y) = 1\,.$$

The uniqueness of the limit implies that the limit of f at $(0,0)$ does not exist.