微積分 MA1002-A 上課筆記（精簡版）

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Definition 9.98

If a function f has derivatives of all orders at $x = c$, then the series

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k$$

is called the Taylor series for f at c. It is also called the Maclaurin series for f if $c = 0$.

Theorem 9.99

Let f be a function that has derivatives of all orders at $x = c$, and P_n be the n-th Taylor polynomial for f at c. If R_n, the remainder associated with P_n, has the property that

$$\lim_{n\to\infty} R_n(x) = 0 \qquad \forall\, x \in I$$

for some interval I, then the Taylor series for f converges and equals $f(x)$; that is,

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \qquad \forall\, x \in I\,.$$

Corollary 9.100

Let f be a function that has derivatives of all orders in an open interval I. If there exists $M > 0$ such that $\left|f^{(k)}(x)\right| \leqslant M$ for all $x \in I$ and each $k \in \mathbb{N}$, then

$$f(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(c)}{k!}(x-c)^k \qquad \forall\, x \in I\,.$$

Example 9.101. Since the k-th derivatives of the sine function is bounded by 1; that is,

$$\left|\frac{d^k}{dx^k}\sin x\right| \leqslant 1 \qquad \forall\, x \in \mathbb{R} \text{ and } k \in \mathbb{N}\,,$$

Corollary 9.100 implies that for all $c \in \mathbb{R}$,

$$\sin x = \sum_{k=0}^{\infty} \frac{1}{k!}\sin\left(c + \frac{k\pi}{2}\right)(x-c)^k \qquad \forall\, x \in \mathbb{R}\,,$$

here we have used $\dfrac{d^k}{dx^k}\sin x = \sin\left(x + \dfrac{k\pi}{2}\right)$ to compute the k-th derivative of the sine function. In particular,

$$\sin x = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots \qquad \forall\, x \in \mathbb{R}\,.$$

Similarly, for all $c \in \mathbb{R}$,

$$\cos x = \sum_{k=0}^{\infty} \frac{1}{k!} \cos\Big(c + \frac{k\pi}{2}\Big)(x-c)^k \qquad \forall\, x \in \mathbb{R}\,.$$

Example 9.102. Consider the natural exponential function $y = \exp(x)$. Note that for all real numbers $R > 0$, we have

$$\Big|\frac{d^k}{dx^k} e^x\Big| = e^x \leqslant e^R \qquad \forall\, x \in (-R, R) \text{ and } k \in \mathbb{N}\,;$$

thus Example 9.70 and Corollary 9.100 imply that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots \qquad \forall\, x \in (-R, R)\,.$$

Since the identity above holds for all $R > 0$, we conclude that

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \cdots \qquad \forall\, x \in \mathbb{R}\,.$$

Example 9.103 (Binomial Series)**.** In this example we consider the Maclaurin series, called the binomial series, of the function $f(x) = (1+x)^\alpha$, where $\alpha \in \mathbb{R}$ and $\alpha \neq \mathbb{N} \cup \{0\}$.

We compute the derivative of f and find that

$$\frac{d^k}{dx^k}(1+x)^\alpha = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)(1+x)^{\alpha-k}\,.$$

Therefore,

$$f^{(k)}(0) = \frac{d^k}{dx^k}\Big|_{x=0}(1+x)^\alpha = \alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)$$

and the Maclaurin series for f is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=0}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!} x^k\,.$$

To see the radius of convergence of the Maclaurin series above, we use the ratio test and find that

$$\lim_{n\to\infty} \frac{\dfrac{\big|\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-(n+1)+1)\big|}{(n+1)!}|x|^{n+1}}{\dfrac{\big|\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n+1)\big|}{n!}|x|^{n}} = \lim_{n\to\infty} \frac{|\alpha-n|}{n+1}|x| = |x|\,;$$

thus the radius of convergence of the power series $\sum\limits_{k=0}^{\infty} \dfrac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}x^k$ is 1. Moreover, by Taylor's theorem, for each $x \in (-1,1)$ there exists ξ between 0 and x such that

$$(1+x)^\alpha = \sum_{k=0}^{n} \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}x^k + R_n(x)\,,$$

where

$$R_n(x) = \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n)}{(n+1)!}(1+\xi)^{\alpha-n-1}x^{n+1}\,.$$

Similar to Example 9.74, we have

$$\big|R_n(x)\big| \leqslant \frac{\big|\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-n)\big|}{(n+1)!}x^\alpha \qquad \forall\, x \in (0,1);$$

thus (without detail reasoning) we find that

$$\lim_{n\to\infty} R_n(x) = 0 \qquad \forall\, x \in (0,1)\,.$$

Therefore,

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}x^k \qquad \forall\, x \in (0,1)\,.$$

In fact,

$$(1+x)^\alpha = \sum_{k=0}^{\infty} \frac{\alpha(\alpha-1)(\alpha-2)\cdots(\alpha-k+1)}{k!}x^k \qquad \forall\, x \in (-1,1)\,.$$