微積分 MA1002-A 上課筆記（精簡版）

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- **Lagrange form of the remainder**

Theorem 9.76: Taylor's Theorem

Let $f : (a, b) \to \mathbb{R}$ be $(n+1)$-times differentiable, and $c \in (a, b)$. Then for each $x \in (a, b)$, there exists ξ between x and c such that

$$f(x) = f(c) + f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n + R_n(x)\,, \quad (9.7.1)$$

where the Lagrange form of the remainder $R_n(x)$ is given by

$$R_n(x) = \frac{f^{(n+1)}(\xi)}{(n+1)!}(x-c)^{n+1}\,.$$

Example 9.78. In this example we show that

$$\ln(1+x) = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}x^k}{k} = x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots + \frac{(-1)^{n-1}x^n}{n} + \cdots \quad \forall\, x \in (0, 1]\,. \quad (9.7.2)$$

Note that Taylor's Theorem implies that for all $x > -1$, there exists ξ between 0 and x such that the remainder associated with $P_n(x) = \sum\limits_{k=1}^{n} \frac{(-1)^{k-1}x^k}{k}$ is given by

$$R_n(x) = \frac{(-1)^n}{n+1}(1+\xi)^{-n-1}x^{n+1}\,.$$

Note that since ξ is between 0 and x, we always have

$$0 < \frac{x}{1+\xi} < 1 \qquad \forall\, x \in (0, 1]\,;$$

thus $|R_n(x)| \leqslant \frac{1}{n+1}$ for all $x \in (0, 1]$ and (9.7.2) is concluded because

$$\lim_{n\to\infty} \big|R_n(x)\big| = 0 \qquad \forall\, x \in (0, 1]\,.$$

Example 9.79. In this example we compute $\ln 2$. Note that using (9.7.2) we find that

$$\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{(-1)^{n-1}}{n} + R_n(1)\,,$$

where

$$R_n(1) = \frac{1}{(n+1)!}\Big(\frac{d^{n+1}}{dx^{n+1}}\Big|_{x=\xi} \ln(1+x)\Big)1^{n+1} = \frac{(-1)^n}{n+1}(1+\xi)^{-(n+1)}$$

for some ξ between 0 and 1. Since ξ could be very closed to 0, in this case the best we can estimate $R_n(1)$ is

$$\left|R_n(1)\right| \leqslant \frac{1}{n+1}\,.$$

Therefore, to evaluate $\ln 2$ accurate to eight decimal point, it is required that $n = 10^8$.

Let $c = \frac{e}{2} \approx 1.359140914$. Then

$$\ln c = \ln\big(1 + (c-1)\big) = (c-1) - \frac{(c-1)^2}{2} + \cdots + \frac{(-1)^{n-1}}{n}(c-1)^n + R_n(c-1)\,,$$

where $R_n(c-1)$ is given by

$$R_n(c-1) = \frac{1}{(n+1)!}\Big(\frac{d^{n+1}}{dx^{n+1}}\Big|_{x=\xi}\ln(1+x)\Big)(c-1)^{n+1} = \frac{(-1)^n}{n+1}(1+\xi)^{-(n+1)}(c-1)^{n+1}$$

for some ξ between 0 and $c-1$. Note that

$$\left|R_n(c)\right| \leqslant \frac{(c-1)^{n+1}}{n+1}\,;$$

thus the value

$$(c-1) - \frac{(c-1)^2}{2} + \frac{(c-1)^3}{3} - \frac{(c-1)^4}{4} + \cdots + \frac{1}{17}(c-1)^{17}$$

to approximate $\ln c$ is accurate to eight decimal points $\big($since $\frac{1}{18}0.4^{18} < 10^{-8}\big)$. On the other hand, we have $\ln 2 = 1 - \ln c$, so the value

$$1 - (c-1) + \frac{(c-1)^2}{2} - \frac{(c-1)^3}{3} + \frac{(c-1)^4}{4} + \cdots - \frac{1}{17}(c-1)^{17}$$

to approximate $\ln 2$ is also accurate to eight decimal points.

9.8 Power Series

Recall that for all $x \in \mathbb{R}$, we have shown that

$$\begin{aligned}
e^x &= \sum_{k=0}^{\infty}\frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots\,,\\
\cos x &= \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k)!}x^{2k} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{(-1)^n}{(2n)!}x^{2n} + \cdots\,,\\
\sin x &= \sum_{k=0}^{\infty}\frac{(-1)^k}{(2k+1)!}x^{2k+1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^n}{(2n+1)!}x^{2n+1} + \cdots\,.
\end{aligned}$$

The identities above show that the functions $y = \exp(x)$, $y = \cos x$, $y = \sin x$ an be defined using series whose terms are multiples of monomials of x. These kind of series are called power series. To be more precise, we have the following

Definition 9.80: Power Series

Let c be a real number. A power series (of one variable x) centered at c is an infinite series of the form

$$\sum_{k=0}^{\infty} a_k (x-c)^k = a_0 + a_1(x-c)^1 + a_2(x-c)^2 + \cdots ,$$

where a_k is independent of x and represents the coefficient of the k-th term.

Theorem 9.81

Let $\{a_k\}_{k=0}^{\infty}$ be a sequence of real numbers. If $\sum\limits_{k=0}^{\infty} a_k d^k$ converges, then $\sum\limits_{k=0}^{\infty} a_k (x-c)^k$ converges absolutely for all $x \in (c-|d|, c+|d|)$.

Proof. First we note that since $\sum\limits_{k=0}^{\infty} a_k d^k$ converges, $\lim\limits_{n\to\infty} a_n d^n = 0$; thus the boundedness of convergent sequence implies that there exists $M > 0$ such that

$$|a_n d^n| \leqslant M \qquad \forall\, n \in \mathbb{N}\,.$$

Suppose that $|x-c| < |d|$. Then there exists $\varepsilon > 0$ such that $|x-c| < |d| - \varepsilon$. Then

$$|a_n||x-c|^n = |a_n||d|^n \frac{|x-c|^n}{(|d|-\varepsilon)^n} \Big(\frac{|d|-\varepsilon}{|d|}\Big)^n \leqslant M\Big(\frac{|d|-\varepsilon}{|d|}\Big)^n .$$

Therefore, by the convergence of geometric series with ratio between -1 and 1, the direct comparison test implies that the series $\sum\limits_{n=0}^{\infty} a_n (x-c)^n$ converges absolutely. □

Corollary 9.82

For a power series centered at c, precisely one of the following is true.

1. The series converges only at c.
2. There exists $R > 0$ such that the series converges absolutely for $|x-c| < R$ and diverges for $|x-c| > R$.
3. The series converges absolutely for all x.

Definition 9.83: Radius of Convergence and Interval of Convergence

Let a power series centered at c be given. If the power series converges only at c, we say that the radius of convergence of the power series is 0. If the power series converges for $|x - c| < R$ but diverges for $|x - c| > R$, we say that the radius of convergence of the power series is R. If the power series converges for all x, we say that the radius of converges of the power series is ∞. The set of all values of x for which the power series converges is called the interval of convergence of the power series.

Remark 9.84. The radius of convergence of a power series centered at c is the greatest lower bound of the set

$$\{r > 0 \,|\, \text{there exists } x \in (c - r, c + r) \text{ such that the power series diverges}\}\,.$$

Example 9.85. Consider the power series $\sum_{k=0}^{\infty} k!x^k$. Note that for each $x \neq 0$,

$$\lim_{n\to\infty} \frac{|(n+1)!x^{n+1}|}{|n!x^n|} = \lim_{n\to\infty} (n+1)|x| = \infty\,;$$

thus the ratio test implies that the power series $\sum_{k=0}^{\infty} k!x^k$ diverges for all $x \neq 0$. Therefore, the radius of convergence of $\sum_{k=0}^{\infty} k!x^k$ is 0, and the interval of convergence of $\sum_{k=0}^{\infty} k!x^k$ is $\{0\}$.

Example 9.86. Consider the power series $\sum_{k=0}^{\infty} 3(x-2)^k$. Note that for each $x \in \mathbb{R}$,

$$\lim_{n\to\infty} \sqrt[n]{3|x-2|^n} = \lim_{n\to\infty} 3^{\frac{1}{n}}|x-2| = |x-2|\,;$$

thus the ratio test implies that the power series $\sum_{k=0}^{\infty} 3(x-2)^k$ converges absolutely if $|x-2| < 1$ and diverges if $|x-2| > 1$. Therefore, the radius of convergence is 1.

To see the interval of convergence, we still need to determine if the power series converges at end-point 1 or 3. However, the power series clearly does not converge at 1 and 3; thus the interval of convergence is $(1, 3)$.

Example 9.87. Consider the power series $\sum_{k=1}^{\infty} \frac{x^k}{k^2}$. Note that for each $x \in \mathbb{R}$,

$$\lim_{k\to\infty} \frac{\left|\frac{x^{k+1}}{(k+1)^2}\right|}{\left|\frac{x^k}{k^2}\right|} = \lim_{k\to\infty} \frac{k^2|x|}{(k+1)^2} = |x|\,;$$

thus the ratio test implies that the power series $\sum_{k=0}^{\infty} \frac{x^k}{k^2}$ converges absolutely if $|x| < 1$ and diverges if $|x| > 1$. Therefore, the radius of convergence is 1.

To see the interval of convergence, we note that $\sum_{k=1}^{\infty} \frac{1}{k^2}$ converges since it is a p-series with $p = 2$, and $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^2}$ converges since it converges absolutely (or simply because it is an alternating series). Therefore, the interval of convergence of the power series is $[-1, 1]$.