# 微積分 MA1002-A 上課筆記（精簡版）

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## 9.7 Taylor Polynomials and Approximations

Suppose that $f:(a,b)\to\mathbb{R}$ is $(n+1)$-times continuously differentiable; that is, $\dfrac{d^kf}{dx^k}$ is continuous on $(a,b)$ for $1\leqslant k\leqslant n+1$, then for $x\in(a,b)$, the Fundamental Theorem of Calculus and integration-by-parts imply that

$$\begin{aligned}
f(x)-f(c) &= \int_c^x f'(t)\,dt = f'(t)(t-x)\Big|_{t=c}^{t=x} - \int_c^x f''(t)(t-x)\,dt \\
&= -f'(c)(c-x) - \int_c^x f''(t)(t-x)\,dt \\
&= f'(c)(x-c) - \Big[f''(t)\frac{(t-x)^2}{2}\Big|_{t=c}^{t=x} - \int_c^x f'''(t)\frac{(t-x)^2}{2}\,dt\Big] \\
&= f'(c)(x-c) - \Big[-\frac{f''(c)}{2}(c-x)^2 - \int_c^x f'''(t)\frac{(t-x)^2}{2}\,dt\Big] \\
&= f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \int_c^x f'''(t)\frac{(t-x)^2}{2}\,dt \\
&= \cdots\cdots \\
&= f'(c)(x-c) + \frac{f''(c)}{2}(x-c)^2 + \cdots + \frac{f^{(n)}(c)}{n!}(x-c)^n \\
&\quad + (-1)^n\int_c^x f^{(n+1)}(t)\frac{(t-x)^n}{n!}\,dt\,,
\end{aligned}$$

where the last equality can be shown by induction. Therefore,

$$f(x) = \sum_{k=0}^n \frac{f^{(k)}(c)}{k!}(x-c)^k + (-1)^n\int_c^x f^{(n+1)}(t)\frac{(t-x)^n}{n!}\,dt\,. \tag{9.7.1}$$

**Definition 9.69**

If $f$ has $n$ derivatives at $c$, then the polynomial

$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(c)}{k!}(x-c)^k$$

is called the $n$-th (order) Taylor polynomial for $f$ at $c$. The $n$-th Taylor polynomial for $f$ at 0 is also called the $n$-th (order) Maclaurin polynomial for $f$.

**Example 9.70.** The $n$-th Maclaurin polynomial for the function $f(x)=e^x$ is

$$P_n(x) = \sum_{k=0}^n \frac{f^{(k)}(0)}{k!}x^k = \sum_{k=0}^n \frac{1}{k!}x^k = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!}\,.$$

**Example 9.71.** The $n$-th Maclaurin polynomial for the function $f(x) = \ln(1+x)$ is given by

$$\begin{aligned} P_n(x) &= \sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=1}^{n} \frac{f^{(k)}(0)}{k!} x^k = \sum_{k=1}^{n} \frac{(-1)^{k-1}(k-1)!}{k!} x^k = \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} x^k \\ &= x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots + \frac{(-1)^{n-1}}{n} x^n , \end{aligned}$$

here we have used $g^{(k)}(x) = (-1)^{k-1}(k-1)!(x+1)^{-k}$ to compute $g^{(k)}(0)$.

The $n$-th Taylor polynomial for the function $g(x) = \ln x$ at 1 is given by

$$\begin{aligned} Q_n(x) &= \sum_{k=0}^{n} \frac{g^{(k)}(1)}{k!} (x-1)^k = \sum_{k=1}^{n} \frac{g^{(k)}(1)}{k!} (x-1)^k = \sum_{k=1}^{n} \frac{(-1)^{k-1}(k-1)!}{k!} (x-1)^k \\ &= \sum_{k=1}^{n} \frac{(-1)^{k-1}}{k} (x-1)^k \\ &= (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \cdots + \frac{(-1)^{n-1}}{n} (x-1)^n , \end{aligned}$$

here we have used $g^{(k)}(x) = (-1)^{k-1}(k-1)! x^{-k}$ to compute $g^{(k)}(1)$. We note that $Q_n(x) = P_n(x-1)$ (and $g(x) = f(x-1)$).

**Example 9.72.** The $(2n)$-th Maclaurin polynomial for the function $f(x) = \cos x$ is given by

$$\begin{aligned} P_{2n}(x) &= \sum_{k=0}^{2n} \frac{f^{(k)}(0)}{k!} x^k = 1 + \sum_{k=1}^{2n} \frac{f^{(k)}(0)}{k!} x^k = 1 + \sum_{k=1}^{n} \frac{f^{(2k-1)}(0)}{(2k-1)!} x^{2k-1} + \sum_{k=1}^{n} \frac{f^{(2k)}(0)}{(2k)!} x^{2k} \\ &= 1 + \sum_{k=1}^{n} \frac{f^{(2k)}(0)}{(2k)!} x^{2k} = 1 - \frac{x^2}{2} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots + \frac{(-1)^n}{(2n)!} x^{2n} , \end{aligned}$$

here we have used $f^{(k)}(x) = \cos\left(x + \frac{k\pi}{2}\right)$ to compute $f^{(k)}(0)$. We also note that $P_{2n}(x) = P_{2n+1}(x)$ for all $n \in \mathbb{N}$.

The $(2n-1)$-th Maclaurin polynomial for the function $g(x) = \sin x$ is given by

$$\begin{aligned} Q_{2n-1}(x) &= \sum_{k=0}^{2n-1} \frac{g^{(k)}(0)}{k!} x^k = \sum_{k=1}^{2n-1} \frac{g^{(k)}(0)}{k!} x^k = \sum_{k=1}^{n} \frac{g^{(2k-1)}(0)}{(2k-1)!} x^{2k-1} + \sum_{k=1}^{n} \frac{g^{(2k)}(0)}{(2k)!} x^{2k} \\ &= \sum_{k=1}^{n} \frac{g^{(2k-1)}(0)}{(2k-1)!} x^{2k-1} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots + \frac{(-1)^{n-1}}{(2n-1)!} x^{2n-1} , \end{aligned}$$

here we have used $g^{(k)}(x) = \sin\left(x + \frac{k\pi}{2}\right)$ to compute $g^{(k)}(0)$. We also note that $Q_{2n-1}(x) = Q_{2n}(x)$ for all $n \in \mathbb{N}$.

**Remark 9.73.** Using the Maclaurin polynomial given in Example 9.70 and 9.72, conceptually we can explain why the Euler identity $e^{i\theta} = \cos\theta + i\sin\theta$. Note that the $(2n)$-th Maclaurin polynomial for exp, cos, sin are

$$\begin{aligned} P^e_{2n}(x) &= 1 + x + \frac{x^2}{2!} + \cdots + \frac{x^{2n}}{(2n)!}\,, \\ P^c_{2n}(x) &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} + \cdots + \frac{(-1)^n}{(2n)!}x^{2n}\,, \\ P^s_{2n}(x) &= x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots + \frac{(-1)^{n-1}}{(2n-1)!}x^{2n-1}\,. \end{aligned}$$

Substitution $x = i\theta$, we find that

$$P^e_{2n}(i\theta) = P^c_{2n}(\theta) + iP^s_{2n}(\theta) \qquad \forall\,\theta \in \mathbb{R}\,.$$

### 9.7.1 Remainder of Taylor Polynomials

To measure the accuracy of approximating a function value $f(x)$ by the Taylor polynomial, we look for the difference $R_n(x) \equiv f(x) - P_n(x)$, where $P_n$ is the $n$-th Taylor polynomial for $f$ (centered at a certain number $c$). The function $R_n$ is called the remainder associated with the approximation $P_n$.

- **Integral form of the remainder**

Suppose that $f : (a, b) \to \mathbb{R}$ is $(n+1)$-times continuously differentiable, and $c, x \in (a, b)$. By (9.7.1), we find that if $P_n$ is the $n$-th Taylor polynomial for $f$ at $c$, then

$$R_n(x) = (-1)^n \int_c^x f^{(n+1)}(t)\frac{(t-x)^n}{n!}\,dt\,. \tag{9.7.2}$$

**Example 9.74.** Consider the function $f(x) = \exp(x) = e^x$. If $P_n$ is the $n$-th Maclaurin polynomial for $f$, the remainder $R_n$ associated with $P_n$ is given by

$$R_n(x) = (-1)^n \int_0^x f^{(n+1)}(t)\frac{(t-x)^n}{n!}\,dt = (-1)^n \int_0^x e^t\frac{(t-x)^n}{n!}\,dt\,.$$

Therefore, if $x > 0$,

$$\Big|e^x - \sum_{k=0}^n \frac{x^k}{k!}\Big| = \Big|\int_0^x e^t\frac{(t-x)^n}{n!}\,dt\Big| \leqslant \int_0^x e^t\frac{(x-t)^n}{n!}\,dt \leqslant \int_0^x e^x\frac{x^n}{n!}\,dt = \frac{e^x x^{n+1}}{n!}\,. \tag{9.7.3}$$

Note that for each $x > 0$, the series $\sum\limits_{k=0}^{\infty} e^x \frac{x^{n+1}}{n!}$ converges since

$$\lim_{n\to\infty} \frac{e^x \dfrac{x^{(n+1)+1}}{(n+1)!}}{e^x \dfrac{x^{n+1}}{n!}} = \lim_{n\to\infty} \frac{x}{n+1} = 0\,;$$

thus the $n$-th term test shows that $\lim\limits_{n\to\infty} e^x \frac{x^{n+1}}{n!} = 0$. Therefore, for each $x > 0$,

$$\lim_{n\to\infty} \left| e^x - \sum_{k=0}^{n} \frac{x^k}{k!} \right| = 0$$

or equivalently,

$$e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots + \frac{x^n}{n!} + \cdots .$$

In particular, if $x = 1$, (9.7.3) implies that

$$\left| e - \sum_{k=0}^{n} \frac{1}{k!} \right| \leqslant \frac{e}{n!}\,;$$

thus $\left| e - \sum\limits_{k=0}^{17} \frac{1}{k!} \right| < 10^{-8}$.