微積分 MA1002-A 上課筆記（精簡版）

2019.03.19.

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Definition 9.22

The series $\sum_{k=1}^{\infty} a_k$ is said to converge to S if the sequence of the partial sum, denoted by $\{S_n\}_{n=1}^{\infty}$ and defined by

$$S_n \equiv \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n,$$

converges to S. S_n is called the n-th partial sum of the series $\sum_{k=1}^{\infty} a_k$.

When the series converges, we write $S = \sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} a_k$ is said to be convergent.

If $\{S_n\}_{n=1}^{\infty}$ diverges, the series is said to be divergent or diverge. If $\lim_{n\to\infty} S_n = \infty$ (or $-\infty$), the series is said to diverge to ∞ (or $-\infty$).

Theorem 9.27: Cauchy Criteria

A series $\sum_{k=1}^{\infty} a_k$ converges if and only if for every $\varepsilon > 0$, there exists $N > 0$ such that

$$\Big|\sum_{k=n}^{n+p} a_k\Big| < \varepsilon \qquad \text{whenever} \quad n \geqslant N, p \geqslant 0\,.$$

Theorem 9.31

Let $f : [1,\infty) \to \mathbb{R}$ be a non-negative continuous decreasing function. The series $\sum_{k=1}^{\infty} f(k)$ converges if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ converges.

Theorem 9.37

Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers, and $0 \leqslant a_n \leqslant b_n$ for all $n \in \mathbb{N}$.

1. If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.
2. If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} a_k$ diverges.

Theorem 9.42

Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers, $a_n, b_n > 0$ for all $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L\,,$$

where L is a non-zero real number. Then $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} b_k$ converges.

Theorem 9.46: Ratio Test

Let $\sum_{k=1}^{\infty} a_k$ be a series with positive terms.

1. The series $\sum_{k=1}^{\infty} a_k$ converges if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$.
2. The series $\sum_{k=1}^{\infty} a_k$ diverges (to ∞) if $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$.

Theorem 9.51: Root Test

Let $\sum_{k=1}^{\infty} a_k$ be a series with positive terms.

1. The series $\sum_{k=1}^{\infty} a_k$ converges if $\lim_{n\to\infty} \sqrt[n]{a_n} < 1$.
2. The series $\sum_{k=1}^{\infty} a_k$ diverges (to ∞) if $\lim_{n\to\infty} \sqrt[n]{a_n} > 1$.

9.6 Absolute and Conditional Convergence

In the previous three sections we consider the convergence of series whose terms do not have different signs. How about the convergence of series like

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k^p}, \qquad \sum_{k=1}^{\infty} \frac{\sin k}{k^p} \qquad \text{and etc.}$$

In the following two sections, we will focus on how to judge the convergence of a series that has both positive and negative terms.

Definition 9.57

An infinite series $\sum_{k=1}^{\infty} a_k$ is said to be absolutely convergent or converge absolutely if the series $\sum_{k=1}^{\infty} |a_k|$ converges. An infinite series $\sum_{k=1}^{\infty} a_k$ is said to be conditionally convergent or converge conditionally if $\sum_{k=1}^{\infty} a_k$ converges but $\sum_{k=1}^{\infty} |a_k|$ diverges (to ∞).

Example 9.58. The series $\sum_{k=1}^{\infty} \frac{(-1)^k}{k^p}$ converge absolutely for $p > 1$ but does not converge absolutely for $p \leqslant 1$ since the p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p > 1$ and diverges for $p \leqslant 1$.

Example 9.59. The series $\sum_{k=1}^{\infty} \frac{\sin k}{k^p}$ converges absolutely for $p > 1$ since

$$0 \leqslant \left|\frac{\sin n}{n^p}\right| \leqslant \frac{1}{n^p} \qquad \forall\, n \in \mathbb{N}$$

and the p-series $\sum_{k=1}^{\infty} \frac{1}{k^p}$ converges for $p > 1$.

Theorem 9.60

An absolutely convergent series is convergent.（絕對收斂則收斂）

Proof. Let $\sum_{k=1}^{\infty} a_k$ be an absolutely convergent series, and $\varepsilon > 0$ be given. Since $\sum_{k=1}^{\infty} |a_k|$ converges, the Cauchy criteria implies that there exists $N > 0$ such that

$$\left|\sum_{k=n}^{n+p} |a_k|\right| < \varepsilon \qquad \text{whenever } n \geqslant N \text{ and } p \geqslant 0\,.$$

Therefore, if $n \geqslant N$ and $p \geqslant 0$,

$$\left|\sum_{k=n}^{n+p} a_k\right| \leqslant \sum_{k=n}^{n+p} |a_k| < \varepsilon$$

thus the Cauchy criteria implies that $\sum_{k=1}^{\infty} a_k$ converges. □

Corollary 9.61: Ratio and Root Tests

The series $\sum_{k=1}^{\infty} a_k$ converges if $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} < 1$ or $\lim_{n\to\infty} \sqrt[n]{|a_n|} < 1$.

Example 9.62. The series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} k!}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k+1)}$ converges since

$$\lim_{n\to\infty} \frac{\left|\frac{(-1)^{n+2}(n+1)!}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n+3)}\right|}{\left|\frac{(-1)^{n+1} n!}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n+1)}\right|} = \lim_{n\to\infty} \frac{\frac{(n+1)!}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n+3)}}{\frac{n!}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2n+1)}} = \lim_{n\to\infty} \frac{n+1}{2n+3} = \frac{1}{2} < 1$$

which shows the absolute convergence of the series $\sum_{k=1}^{\infty} \frac{(-1)^{k+1} k!}{1 \cdot 3 \cdot 5 \cdot \cdots \cdot (2k+1)}$.

9.6.1 Alternating Series

In the previous two sections we consider the convergence of series whose terms do not have different signs. How about the convergence of series like

$$\sum_{k=1}^{\infty} \frac{(-1)^{k+1}}{k}, \quad \sum_{k=1}^{\infty} \frac{\sin k}{k} \quad \text{and etc.}$$

In the following two sections, we will focus on how to judge the convergence of a series that has both positive and negative terms.

Theorem 9.63: Abel's Test

Let $\{a_n\}_{n=1}^{\infty}$, $\{p_n\}_{n=1}^{\infty}$ be sequences of real numbers such that

1. the sequence of partial sums of the series $\sum\limits_{k=1}^{\infty} a_k$ is bounded; that is, there exists $M \in \mathbb{R}$ such that $\Big|\sum\limits_{k=1}^{n} a_k\Big| \leqslant M$ for all $n \in \mathbb{N}$.
2. $\{p_n\}_{n=1}^{\infty}$ is a decreasing sequence, and $\lim\limits_{n\to\infty} p_n = 0$.

Then $\sum\limits_{k=1}^{\infty} a_k p_k$ converges.

Proof. Let $\varepsilon > 0$ be given. Since $\{p_n\}_{n=1}^{\infty}$ is decreasing and $\lim\limits_{n\to\infty} p_n = 0$, there exists $N > 0$ such that

$$0 \leqslant p_n < \frac{\varepsilon}{2M+1} \qquad \text{whenever} \quad n \geqslant N\,.$$

Define $S_n = \sum\limits_{k=1}^{n} a_k$. Then if $n \geqslant N$ and $\ell \geqslant 0$,

$$\begin{aligned}
\Big|\sum_{k=n}^{n+\ell} a_k p_k\Big| &= \big|(S_n - S_{n-1})p_n + (S_{n+1} - S_n)p_{n+1} + (S_{n+2} - S_{n+1})p_{n+2} + \cdots \\
&\quad + (S_{n+\ell-1} - S_{n+\ell-2})p_{n+\ell-1} + (S_{n+\ell} - S_{n+\ell-1})p_{n+\ell}\big| \\
&= \big|-S_{n-1}p_n + S_n(p_n - p_{n+1}) + S_{n+1}(p_{n+1} - p_{n+2}) + \cdots + S_{n+\ell-1}(p_{n+\ell-1} - p_{n+\ell}) \\
&\quad + S_{n+\ell}p_{n+\ell}\big| \\
&\leqslant |S_{n-1}p_n| + |S_n(p_n - p_{n+1})| + |S_{n+1}(p_{n+1} - p_{n+2})| + \cdots + |S_{n+\ell}(p_{n+\ell-1} - p_{n+\ell})| \\
&\quad + |S_{n+\ell+1}p_{n+\ell}| \\
&\leqslant Mp_n + M(p_n - p_{n+1}) + M(p_{n+1} - p_{n+2}) + \cdots + M(p_{n+\ell-1} - p_{n+\ell}) + Mp_{n+\ell} \\
&= 2Mp_n < \frac{2M\varepsilon}{2M+1} < \varepsilon\,.
\end{aligned}$$

By the Cauchy criteria (Theorem 9.27), $\sum\limits_{k=1}^{\infty} a_k p_k$ converges. □

Corollary 9.64

Let $\{p_n\}_{n=1}^{\infty}$ be a decreasing sequence of real numbers. If $\lim\limits_{n\to\infty} p_n = 0$, then $\sum\limits_{k=1}^{\infty} (-1)^k p_k$ and $\sum\limits_{k=1}^{\infty} (-1)^{k+1} p_k$ converge.

Example 9.65. The series $\sum\limits_{k=1}^{\infty} \dfrac{(-1)^{k+1}}{k^p}$ converges conditionally for $0 < p \leqslant 1$ since

1. $\sum\limits_{k=1}^{\infty} \dfrac{(-1)^{k+1}}{k^p}$ converges due the fact that

$$\Big|\sum_{k=1}^{n} (-1)^{k+1}\Big| \leqslant 1 \quad \text{and} \quad \Big\{\frac{1}{n^p}\Big\}_{n=1}^{\infty} \text{ is decreasing and converges to } 0\,.$$

2. $\sum\limits_{k=1}^{\infty} \Big|\dfrac{(-1)^{k+1}}{k^p}\Big|$ diverges for it is a p-series with $p = 1$.

Similarly, $\sum\limits_{k=1}^{\infty} \dfrac{(-1)^k}{\ln(k+1)}$ converges conditionally.

Example 9.66. The series $\sum\limits_{k=1}^{\infty} \dfrac{\sin k}{k^p}$ converges for $p > 0$ since

1. $\sum\limits_{k=1}^{n} \sin k = \dfrac{\cos\frac{1}{2} - \cos\frac{2k+1}{2}}{2\sin\frac{1}{2}}$; $\Big(\text{thus } \Big|\sum\limits_{k=1}^{n} \sin k\Big| \leqslant \dfrac{1}{\sin\frac{1}{2}}\Big)$.

2. $\{\dfrac{1}{n^p}\}_{n=1}^{\infty}$ is decreasing and $\lim\limits_{n\to\infty} \dfrac{1}{n^p} = 0$.

We remark here that $\sum\limits_{k=1}^{\infty} \dfrac{\sin k}{k} = \dfrac{\pi - 1}{2}$. In fact, $\sum\limits_{k=1}^{\infty} \dfrac{\sin(kx)}{k}$ is the Fourier series of the function $\dfrac{\pi - x}{2}$.

- **Alternating Series Remainder**

Theorem 9.67

Let $\{a_n\}_{n=1}^{\infty}$, $\{p_n\}_{n=1}^{\infty}$ be sequences of real numbers satisfying conditions in Theorem 9.63. Then

$$\Big|\sum_{k=1}^{\infty} a_k p_k - \sum_{k=1}^{n} a_k p_k\Big| = \Big|\sum_{k=n+1}^{\infty} a_k p_k\Big| \leqslant 2M p_{n+1} \qquad \forall\, n \in \mathbb{N}\,.$$

Moreover, if $a_k = (-1)^k$, and $S = \sum\limits_{k=1}^{\infty} (-1)^k p_k$ be an alternating series, and S_n be the n-th partial sum of the series, then

$$|S - S_n| \leqslant p_{n+1} \qquad \forall\, n \in \mathbb{N}\,.$$

Example 9.68. Approximate the sum of the series $\sum\limits_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k!}$ by its first six terms, we obtain that

$$\sum_{k=1}^{6} (-1)^{k+1} \frac{1}{k!} = \frac{1}{1!} - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \frac{1}{5!} - \frac{1}{6!} \approx 0.63194\,.$$

Moreover, by Theorem 9.69, we find that

$$\Big|\sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{k!} - \sum_{k=1}^{6} (-1)^{k+1} \frac{1}{k!}\Big| \leqslant \frac{1}{7!} = \frac{1}{5040} \approx 0.0002\,.$$