微積分 MA1002-A 上課筆記（精簡版）

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Definition 9.22

The series $\sum_{k=1}^{\infty} a_k$ is said to converge to S if the sequence of the partial sum, denoted by $\{S_n\}_{n=1}^{\infty}$ and defined by

$$S_n \equiv \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n,$$

converges to S. S_n is called the n-th partial sum of the series $\sum_{k=1}^{\infty} a_k$.

When the series converges, we write $S = \sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} a_k$ is said to be convergent.

If $\{S_n\}_{n=1}^{\infty}$ diverges, the series is said to be divergent or diverge. If $\lim_{n\to\infty} S_n = \infty$ (or $-\infty$), the series is said to diverge to ∞ (or $-\infty$).

Theorem 9.31

Let $f : [1,\infty) \to \mathbb{R}$ be a non-negative continuous decreasing function. The series $\sum_{k=1}^{\infty} f(k)$ converges if and only if the improper integral $\int_1^{\infty} f(x)\,dx$ converges.

Theorem 9.37

Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers, and $0 \leqslant a_n \leqslant b_n$ for all $n \in \mathbb{N}$.

1. If $\sum_{k=1}^{\infty} b_k$ converges, then $\sum_{k=1}^{\infty} a_k$ converges.
2. If $\sum_{k=1}^{\infty} a_k$ diverges, then $\sum_{k=1}^{\infty} a_k$ diverges.

9.4.2 Limit Comparison Test

Theorem 9.42

Let $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers, $a_n, b_n > 0$ for all $n \in \mathbb{N}$, and

$$\lim_{n\to\infty} \frac{a_n}{b_n} = L\,,$$

where L is a non-zero real number. Then $\sum_{k=1}^{\infty} a_k$ converges if and only if $\sum_{k=1}^{\infty} b_k$ converges.

Proof. We first note that if $L \neq 0$, then $L > 0$ since $\dfrac{a_n}{b_n} > 0$ for all $n \in \mathbb{N}$. By the fact that

$\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = L$, there exists $N > 0$ such that

$$\left|\frac{a_n}{b_n} - L\right| < \frac{L}{2} \qquad \text{whenever} \quad n \geqslant N\,.$$

In other words, $\dfrac{L}{2} < \dfrac{a_n}{b_n} < \dfrac{3L}{2}$ for all $n \geqslant N$; thus

$$0 < a_n < \frac{3L}{2} b_n \ \text{ and } \ 0 < b_n < \frac{2}{L} a_n \qquad \text{whenever} \quad n \geqslant N\,.$$

By Theorem 9.37 and Remark 9.38, we find that $\sum\limits_{k=1}^{\infty} a_k$ converges if and only if $\sum\limits_{k=1}^{\infty} b_k$ converges. □

Remark 9.43. 1. If $\lim\limits_{n\to\infty} \dfrac{a_n}{b_n} = 0$, then the convergence of $\sum\limits_{k=1}^{\infty} b_k$ implies the convergence of $\sum\limits_{k=1}^{\infty} a_k$, but not necessary the reverse direction.

2. The condition "$a_n, b_n > 0$ for all $n \in \mathbb{N}$" can be relaxed by "a_n and b_n are sign-definite for $n \geqslant N$, where a sequence $\{c_n\}_{n=1}^{\infty}$ is called sign-definite for $n \geqslant N$ if $c_n > 0$ for all $n \geqslant N$ or $c_n < 0$ for all $n \geqslant N$.

Example 9.44. Recall that in Example 9.40 and 9.41 we have shown that the series $\sum\limits_{k=1}^{\infty} \dfrac{1}{2+3^k}$ converges and the series $\sum\limits_{k=1}^{\infty} \dfrac{1}{2+\sqrt{k}}$ diverges using the direct comparison test. Note that since

$$\lim_{n\to\infty} \frac{\dfrac{1}{2+3^n}}{\dfrac{1}{3^n}} = 1 \quad \text{and} \quad \lim_{n\to\infty} \frac{\dfrac{1}{2+\sqrt{n}}}{\dfrac{1}{\sqrt{n}}} = 1\,,$$

using the convergence of the p-series and the limit comparison test we can also conclude that $\sum\limits_{k=1}^{\infty} \dfrac{1}{2+3^k}$ converges and $\sum\limits_{k=1}^{\infty} \dfrac{1}{2+\sqrt{k}}$ diverges.

Example 9.45. The general harmonic series $\sum\limits_{k=1}^{\infty} \dfrac{1}{ak+b}$ diverges for the following reasons:

1. if $a = 0$, then clearly $\sum\limits_{k=1}^{\infty} \dfrac{1}{b}$ diverges.

2. if $a \neq 0$, then $\sum\limits_{k=1}^{\infty} \dfrac{1}{ak}$ diverges and $\lim\limits_{n\to\infty} \dfrac{\frac{1}{ak}}{\frac{1}{ak+b}} = 1$.

9.5 The Ratio and Root Tests

9.5.1 The Ratio Test

Theorem 9.46: Ratio Test

Let $\sum\limits_{k=1}^{\infty} a_k$ be a series with positive terms.

1. The series $\sum\limits_{k=1}^{\infty} a_k$ converges if $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} < 1$.
2. The series $\sum\limits_{k=1}^{\infty} a_k$ diverges (to ∞) if $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} > 1$.

Proof. Suppose that $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = L$ exists. Define $r = \frac{L+1}{2}$.

1. Assume that $L < 1$. Then for $\varepsilon = \frac{1-L}{2}$, there exists $N > 0$ such that
$$\Big|\frac{a_{n+1}}{a_n} - L\Big| < \frac{1-L}{2} \qquad \text{whenever } n \geqslant N\,;$$
thus
$$0 < \frac{a_{n+1}}{a_n} < r \qquad \text{whenever } n \geqslant N\,.$$
Note that $0 < r < 1$, and the inequality above implies that if $n \geqslant N$, $a_{n+1} < ra_n$. Therefore,
$$0 < a_n \leqslant a_N r^{n-N} \qquad \text{for all } n \geqslant N\,.$$
Now, since the series $\sum\limits_{k=1}^{\infty} a_N r^{k-N}$ converges, the comparison test implies that $\sum\limits_{k=1}^{\infty} a_k$ converges as well.

2. Assume that $L > 1$. Then for $\varepsilon = \frac{L-1}{2}$, there exists $N > 0$ such that
$$\Big|\frac{a_{n+1}}{a_n} - L\Big| < \frac{L-1}{2} \qquad \text{whenever } n \geqslant N\,;$$
thus $r < \frac{a_{n+1}}{a_n}$ whenever $n \geqslant N$. Note that $r > 1$, and the inequality above implies that if $n \geqslant N$, $a_{n+1} > ra_n$. Therefore,
$$0 < a_N r^{n-N} \leqslant a_n \qquad \text{for all } n \geqslant N\,.$$
Now, since the series $\sum\limits_{k=1}^{\infty} a_N r^{k-N}$ diverges, the comparison test implies that $\sum\limits_{k=1}^{\infty} a_k$ diverges as well.

□

Remark 9.47. When $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$, the convergence or divergence of $\sum_{n=1}^{\infty} a_k$ cannot be concluded. For example, the p-series could converge or diverge depending on how large p is, but no matter what p is,

$$\lim_{n\to\infty} \frac{(n+1)^p}{n^p} = 1\,.$$

Example 9.48. The series $\sum_{k=1}^{\infty} \frac{2^k}{k!}$ converges since

$$\lim_{n\to\infty} \frac{2^{n+1}/(n+1)!}{2^n/n!} = \lim_{n\to\infty} \frac{2}{n+1} = 0 < 1\,.$$

Example 9.49. The series $\sum_{k=1}^{\infty} \frac{k^2 2^{k+1}}{3^k}$ converges since

$$\lim_{n\to\infty} \frac{(n+1)^2 2^{n+2}/3^{n+1}}{n^2 2^{n+1}/3^n} = \lim_{n\to\infty} \frac{2}{3}\frac{(n+1)^2}{n^2} = \frac{2}{3} < 1\,.$$

Example 9.50. The series $\sum_{k=1}^{\infty} \frac{k^k}{k!}$ diverges since

$$\lim_{n\to\infty} \frac{(n+1)^{n+1}/(n+1)!}{n^n/n!} = \lim_{n\to\infty} \Big(1 + \frac{1}{n}\Big)^n = e > 1\,.$$

9.5.2 The Root Test

Theorem 9.51: Root Test

Let $\sum_{k=1}^{\infty} a_k$ be a series with positive terms.

1. The series $\sum_{k=1}^{\infty} a_k$ converges if $\lim_{n\to\infty} \sqrt[n]{a_n} < 1$.
2. The series $\sum_{k=1}^{\infty} a_k$ diverges (to ∞) if $\lim_{n\to\infty} \sqrt[n]{a_n} > 1$.

Proof. Suppose that $\lim_{n\to\infty} \sqrt[n]{a_n} = L$ exists. Define $r = \frac{L+1}{2}$.

1. Assume that $L < 1$. Then for $\varepsilon = \frac{1-L}{2}$, there exists $N > 0$ such that

$$\big|\sqrt[n]{a_n} - L\big| < \frac{1-L}{2} \qquad \text{whenever } n \geqslant N\,;$$

thus

$$0 < \sqrt[n]{a_n} < r \qquad \text{whenever } n \geqslant N$$

or equivalently,

$$0 < a_n \leqslant r^n \qquad \text{whenever } n \geqslant N\,.$$

By the fact that $0 < r < 1$, the series $\sum\limits_{k=1}^{\infty} r^k$ converges; thus the comparison test implies that $\sum\limits_{k=1}^{\infty} a_k$ converges as well.

2. Left as an exercise. □

Remark 9.52. When $\lim\limits_{n\to\infty} \sqrt[n]{a_n} = 1$, the convergence or divergence of $\sum\limits_{n=1}^{\infty} a_k$ cannot be concluded. For example, the p-series could converge or diverge depending on how large p is, but no matter what p is,

$$\lim_{n\to\infty} \sqrt[n]{n^p} = \Big(\lim_{n\to\infty} \sqrt[n]{n}\Big)^p = 1\,.$$

Example 9.53. The series $\sum\limits_{k=1}^{\infty} \frac{e^{2k}}{k^k}$ converges since

$$\lim_{n\to\infty} \Big(\frac{e^{2n}}{n^n}\Big)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{e^2}{n} = 0 < 1\,.$$

We also note that the convergence of this series can be obtained through the ratio test:

$$\lim_{n\to\infty} \frac{e^{2(n+1)}/(n+1)^{n+1}}{e^{2n}/n^n} = \lim_{n\to\infty} \frac{e^2}{n+1}\Big(1+\frac{1}{n}\Big)^{-n} = 0 < 1\,.$$

Example 9.54. The series $\sum\limits_{k=1}^{\infty} \frac{k^2 2^{k+1}}{3^k}$ converges since

$$\lim_{n\to\infty} \Big(\frac{n^2 2^{n+1}}{3^n}\Big)^{\frac{1}{n}} = \lim_{n\to\infty} \frac{2(2n^2)^{\frac{1}{n}}}{3} = \frac{2}{3} < 1\,.$$

Example 9.55. The series $\sum\limits_{k=1}^{\infty} \frac{k^k}{k!}$ diverges since

$$\lim_{n\to\infty} \Big(\frac{n^n}{n!}\Big)^{\frac{1}{n}} = \lim_{n\to\infty} \Big(\frac{n^n}{\sqrt{2\pi n}n^n e^{-n}} \frac{\sqrt{2\pi n}n^n e^{-n}}{n!}\Big)^{\frac{1}{n}} = \lim_{n\to\infty} \Big(\frac{e^n}{\sqrt{2\pi n}}\Big)^{\frac{1}{n}} = e > 1\,,$$

here we have used Stirling's formula (9.1.2) to compute the limit.

Remark 9.56. Observe from Example 9.49, 9.50, 9.54 and 9.55, we see that as long as $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n}$ and $\lim\limits_{n\to\infty} \sqrt[n]{a_n}$ exists, then the limits are the same. This is in fact true in general, but we will not prove it since this is not our focus.