微積分 MA1002-A 上課筆記（精簡版）

2019.03.12.

Ching-hsiao Arthur Cheng 鄭經斅

Definition 9.1: Sequence

A ***sequence*** of real numbers (or simply a real sequence) is a function $f : \mathbb{N} \to \mathbb{R}$. We usually use f_n to denote $f(n)$, the n-th term of a sequence $f : \mathbb{N} \to \mathbb{R}$, and this sequence is usually denoted by $\{f_n\}_{n=1}^{\infty}$ or simply $\{f_n\}$.

Definition 9.5

A sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ is said to ***converge to*** L if for every $\varepsilon > 0$, there exists $N > 0$ such that $|a_n - L| < \varepsilon$ whenever $n \geqslant N$. Such an L (must be a real number and) is called a ***limit*** of the sequence. If $\{a_n\}_{n=1}^{\infty}$ converges to L, we write $a_n \to x$ as $n \to \infty$.

A sequence of real number $\{a_n\}_{n=1}^{\infty}$ is said to be ***convergent*** if there exists $L \in \mathbb{R}$ such that $\{a_n\}_{n=1}^{\infty}$ converges to L. If no such L exists we say that $\{a_n\}_{n=1}^{\infty}$ ***does not converge*** or simply ***diverges***.

Proposition 9.6

If $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers, and $a_n \to a$ and $a_n \to b$ as $n \to \infty$, then $a = b$. (若收斂則極限唯一).

• **Notation**: Since the limit of a convergent sequence is unique, we use $\lim\limits_{n\to\infty} a_n$ to denote this unique limit of a convergent sequence $\{a_n\}_{n=1}^{\infty}$.

• **Completeness of Real Numbers**:

One important property of the real numbers is that they are ***complete***. The completeness axiom for real numbers states that "every bounded sequence of real numbers has a ***least upper bound*** and a ***greatest lower bound***"; that is, if $\{a_n\}_{n=1}^{\infty}$ is a bounded sequence of real numbers, then there exists an upper bound M and a lower bound m of $\{a_n\}_{n=1}^{\infty}$ such that there is no smaller upper bound nor greater lower bound of $\{a_n\}_{n=1}^{\infty}$.

Theorem 9.20: Monotone Sequence Property (MSP)

Let $\{a_n\}_{n=1}^{\infty}$ be a monotone sequence of real numbers. Then $\{a_n\}_{n=1}^{\infty}$ converges if and only if $\{a_n\}_{n=1}^{\infty}$ is bounded.

Remark 9.21. A sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ is called a ***Cauchy sequence*** if for every $\varepsilon > 0$ there exists $N > 0$ such that

$$|a_n - a_m| < \varepsilon \qquad \text{whenever} \quad n, m \geqslant N\,.$$

A convergent sequence must be a Cauchy sequence. Moreover, the completeness of real numbers is equivalent to that "every Cauchy sequence of real number converges".

9.2 Series and Convergence

Definition 9.22

The series $\sum_{k=1}^{\infty} a_k$ is said to converge to S if the sequence of the partial sum, denoted by $\{S_n\}_{n=1}^{\infty}$ and defined by

$$S_n \equiv \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n,$$

converges to S. S_n is called the n-th partial sum of the series $\sum_{k=1}^{\infty} a_k$.

When the series converges, we write $S = \sum_{k=1}^{\infty} a_k$ and $\sum_{k=1}^{\infty} a_k$ is said to be convergent.

If $\{S_n\}_{n=1}^{\infty}$ diverges, the series is said to be divergent or diverge. If $\lim_{n\to\infty} S_n = \infty$ (or $-\infty$), the series is said to diverge to ∞ (or $-\infty$).

9.3 The Integral Test and p-Series

9.3.1 The integral test

Suppose that the sequence $\{a_n\}_{n=1}^{\infty}$ is obtained by evaluating a non-negative continuous decreasing function $f : [1, \infty) \to \mathbb{R}$ on $\mathbb{N}$; that is, $f(n) = a_n$. Then

$$\int_1^{n+1} f(x)\, dx \leqslant S_n \equiv \sum_{k=1}^{n} a_k \leqslant a_1 + \int_1^{n} f(x)\, dx\,. \tag{9.3.1}$$

Since the sequence of partial sums $\{S_n\}_{n=1}^{\infty}$ of the series $\sum_{k=1}^{\infty} a_k$ is increasing, the completeness of real numbers implies that $\{S_n\}_{n=1}^{\infty}$ converges if and only if the improper integral $\int_1^{\infty} f(x)\, dx$ converges.

Theorem 9.31

Let $f : [1, \infty) \to \mathbb{R}$ be a non-negative continuous decreasing function. The series $\sum_{k=1}^{\infty} f(k)$ converges if and only if the improper integral $\int_1^{\infty} f(x)\, dx$ converges.

Example 9.32. The series $\sum\limits_{k=1}^{\infty} \frac{1}{k^2+1}$ converges since

$$\int_1^{\infty} \frac{dx}{x^2+1} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^2+1} = \lim_{b\to\infty} \arctan x \Big|_{x=1}^{x=b} = \lim_{b\to\infty} (\arctan b - \arctan 1) = \frac{\pi}{4}$$

and the function $f(x) = \dfrac{1}{x^2+1}$ is non-negative continuous and decreasing on $[1, \infty)$.

Example 9.33. The series $\sum\limits_{k=1}^{\infty} \frac{k}{k^2+1}$ diverges since

$$\int_1^{\infty} \frac{x}{x^2+1}\,dx = \lim_{b\to\infty} \int_1^b \frac{x}{x^2+1}\,dx = \lim_{b\to\infty} \frac{\ln(x^2+1)}{2}\Big|_{x=1}^{x=b} = \frac{1}{2}\lim_{b\to\infty} \big[\ln(b^2+1) - \ln 2\big] = \infty$$

and the function $f(x) = \dfrac{x}{x^2+1}$ is non-negative continuous and decreasing on $[1, \infty)$.

Example 9.34. The series $\sum\limits_{k=2}^{\infty} \frac{1}{k\ln k}$ converges since

$$\begin{aligned}\int_2^{\infty} \frac{dx}{x\ln x} &= \lim_{b\to\infty} \int_2^b \frac{dx}{x\ln x} \overset{(x=e^u)}{=} \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{e^u du}{e^u \ln e^u} = \lim_{b\to\infty} \int_{\ln 2}^{\ln b} \frac{du}{u} = \lim_{b\to\infty} \ln u \Big|_{u=\ln 2}^{u=\ln b} \\ &= \lim_{b\to\infty} (\ln\ln b - \ln\ln 2) = \infty\end{aligned}$$

and the function $f(x) = \dfrac{1}{x\ln x}$ is non-negative continuous and decreasing on $[2, \infty)$.

9.3.2 p-series

A series of the form

$$\sum_{k=1}^{\infty} \frac{1}{k^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots$$

is called a p-series. The series is a function of p, and this function is usually called the ***Riemann zeta function***; that is,

$$\zeta(s) \equiv \sum_{n=1}^{\infty} \frac{1}{n^s}\,.$$

A harmonic series is the p-series with $p = 1$, and a general harmonic series is of the form

$$\sum_{k=1}^{\infty} \frac{1}{ak+b}\,.$$

By Theorem 8.48 and 9.31, the p-series converges if and only if $p > 1$.

Remark 9.35. It can be shown that $\sum\limits_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$. In fact, for all integer $k \geqslant 2$, the number $\sum\limits_{k=1}^{\infty} \frac{1}{n^k}$ can be computed by hand (even though it is very time consuming).

Remark 9.36. Using (9.3.1), we find that

$$\ln(n+1) \leqslant \sum_{k=1}^{n} \frac{1}{k} \leqslant 1 + \ln n \qquad \forall\, n \in \mathbb{N}\,.$$

Therefore, the sequence $\{a_n\}_{n=1}^{\infty}$ defined by

$$a_n = \sum_{k=1}^{n} \frac{1}{k} - \ln n$$

is bounded. Moreover,

$$a_n - a_{n+1} = \sum_{k=1}^{n} \frac{1}{k} - \ln n - \sum_{k=1}^{n+1} \frac{1}{k} + \ln(n+1) = \ln\big(1 + \frac{1}{n}\big) - \frac{1}{n+1}\,.$$

Since the derivative of the function $f(x) = \ln(1+x) - \frac{x}{x+1}$ is positive on $[0,1]$, we find that f is increasing on $[0,1]$; thus

$$\ln\big(1 + \frac{1}{n}\big) - \frac{1}{n+1} = f\big(\frac{1}{n}\big) \geqslant f(0) = \ln 1 - \frac{0}{1} = 0 \qquad \forall\, n \in \mathbb{N}$$

which shows that $a_n \geqslant a_{n+1}$. Therefore, $\{a_n\}_{n=1}^{\infty}$ is monotone decreasing and bounded from below (by 0). The completeness of real numbers then implies the convergence of the sequence $\{a_n\}_{n=1}^{\infty}$. The limit

$$\lim_{n\to\infty} \Big(\sum_{k=1}^{n} \frac{1}{k} - \ln n\Big)$$

is called Euler's constant.

9.4 Comparisons of Series

When the sequence $\{a_n\}_{n=1}^{\infty}$ is not obtained by $a_n = f(n)$ for some decreasing function $f : [1,\infty) \to \mathbb{R}$, the convergence of the series $\sum\limits_{k=1}^{\infty} a_k$ cannot be judged by the convergence of the improper integral $\int_1^{\infty} f(x)\,dx$. To determine the convergence of this kind of series, usually one uses comparison tests.

9.4.1 Direct Comparison Test

Theorem 9.37

Let $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$ be sequences of real numbers, and $0 \leqslant a_n \leqslant b_n$ for all $n \in \mathbb{N}$.

1. If $\sum\limits_{k=1}^\infty b_k$ converges, then $\sum\limits_{k=1}^\infty a_k$ converges.
2. If $\sum\limits_{k=1}^\infty a_k$ diverges, then $\sum\limits_{k=1}^\infty a_k$ diverges.

Proof. Let S_n and T_n be the n-th partial sum of the series $\sum\limits_{k=1}^\infty a_k$ and $\sum\limits_{k=1}^\infty b_k$, respectively; that is,

$$S_n = \sum_{k=1}^n a_k \qquad \text{and} \qquad T_n = \sum_{k=1}^n b_k\,.$$

Then by the assumption that $0 \leqslant a_n \leqslant b_n$ for all $n \in \mathbb{N}$, we find that $0 \leqslant S_n \leqslant T_n$ for all $n \in \mathbb{N}$, and $\{S_n\}_{n=1}^\infty$ and $\{T_n\}_{n=1}^\infty$ are monotone increasing sequences.

1. If $\sum\limits_{k=1}^\infty b_k$ converges, $\lim\limits_{n\to\infty} T_n = T$ exists; thus $0 \leqslant S_n \leqslant T_n \leqslant T$ for all $n \in \mathbb{N}$. Since $\{S_n\}_{n=1}^\infty$ is increasing, the monotone sequence property shows that $\lim\limits_{n\to\infty} S_n$ exists; thus $\sum\limits_{k=1}^\infty a_k$ converges.
2. If $\sum\limits_{k=1}^\infty a_k$ diverges, $\lim\limits_{n\to\infty} S_n = \infty$; thus by the fact that $S_n \leqslant T_n$ for all $n \in \mathbb{N}$, we find that $\lim\limits_{n\to\infty} T_n = \infty$. Therefore, $\sum\limits_{k=1}^\infty b_k$ diverges (to ∞). □

Remark 9.38. It does not require that $0 \leqslant a_n \leqslant b_n$ for all $n \in \mathbb{N}$ for the direct comparison test to hold. The condition can be relaxed by that "$0 \leqslant a_n \leqslant b_n$ for all $n \geqslant N$" for some N since the sum of the first $N-1$ terms does not affect the convergence of the series.

Example 9.39. The series $\sum\limits_{k=1}^\infty \dfrac{1+\sin k}{k^2}$ converges since $\dfrac{1+\sin n}{n^2} \leqslant \dfrac{2}{n^2}$ for all $n \in \mathbb{N}$ and the p-series $\sum\limits_{k=1}^\infty \dfrac{2}{k^2}$ converges.

Example 9.40. The series $\sum\limits_{k=1}^\infty \dfrac{1}{2+3^k}$ converges since $\dfrac{1}{2+3^n} \leqslant \dfrac{1}{3^n}$ for all $n \in \mathbb{N}$ and the geometric series $\sum\limits_{k=1}^\infty \dfrac{1}{3^k}$ converges.

Example 9.41. The series $\sum_{k=1}^{\infty} \frac{1}{2+\sqrt{k}}$ diverges since $\frac{1}{2+\sqrt{n}} \geqslant \frac{1}{3\sqrt{n}}$ for all $n \in \mathbb{N}$ and the p-series $\sum_{k=1}^{\infty} \frac{1}{3\sqrt{k}} = \frac{1}{3} \sum_{k=1}^{\infty} \frac{1}{\sqrt{k}}$ diverges.

One can also use the fact that $\frac{1}{2+\sqrt{n}} \geqslant \frac{1}{n}$ for all $n \geqslant 4$ and $\sum_{k=1}^{\infty} \frac{1}{k}$ diverges to conclude that $\sum_{k=1}^{\infty} \frac{1}{2+\sqrt{k}}$ diverges.