微積分 MA1002-A 上課筆記（精簡版）

2019.03.05.

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Chapter 9

Infinite Series

9.1 Sequences

Definition 9.1: Sequence

A ***sequence*** of real numbers (or simply a real sequence) is a function $f : \mathbb{N} \to \mathbb{R}$. The collection of numbers $\{f(1), f(2), f(3), \cdots\}$ are called ***terms*** of the sequence and the value of f at n is called the ***n-th term*** of the sequence. We usually use f_n to denote the n-th term of a sequence $f : \mathbb{N} \to \mathbb{R}$, and this sequence is usually denoted by $\{f_n\}_{n=1}^{\infty}$ or simply $\{f_n\}$.

Example 9.2. Let $f : \mathbb{N} \to \mathbb{R}$ be the sequence defined by $f(n) = 3 + (-1)^n$. Then f is a real sequence. Its terms are $\{2, 4, 2, 4, \cdots\}$.

Example 9.3. A sequence can also be defined recursively. For example, let $\{a_n\}_{n=1}^{\infty}$ be defined by

$$a_{n+1} = \sqrt{2a_n}\,, \qquad d_1 = \sqrt{2}\,.$$

Then $a_2 = \sqrt{2\sqrt{2}}$, $a_3 = \sqrt{2\sqrt{2\sqrt{2}}}$, and etc. The general form of a_n is given by

$$a_n = 2^{\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\cdots+\frac{1}{2^n}} = 2^{\frac{2^n-1}{2^n}}\,.$$

There are also sequences that are defined recursively but it is difficult to obtain the general form of the sequence. For example, let $\{b_n\}_{n=1}^{\infty}$ be defined by

$$b_{n+1} = \sqrt{2 + b_n}\,, \qquad b_1 = \sqrt{2}\,.$$

Then $b_2 = \sqrt{2+\sqrt{2}}$, $b_3 = \sqrt{2+\sqrt{2+\sqrt{2}}}$, and etc.

Remark 9.4. Occasionally, it is convenient to begin a sequence with the 0-th term or even the k-th term. In such cases, we write $\{a_n\}_{n=0}^{\infty}$ and $\{a_n\}_{n=k}^{\infty}$ to denote the sequences.

The primary goal of this chapter concerns the limit of a sequence; that is, we would like to know to which value the n-th term of a sequence approaches as n become larger and larger.

Definition 9.5

A sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ is said to ***converge to*** L if for every $\varepsilon > 0$, there exists $N > 0$ such that

$$|a_n - L| < \varepsilon \quad \text{whenever} \quad n \geqslant N \,.$$

Such an L (must be a real number and) is called a ***limit*** of the sequence. If $\{a_n\}_{n=1}^{\infty}$ converges to L, we write $a_n \to x$ as $n \to \infty$.
A sequence of real number $\{a_n\}_{n=1}^{\infty}$ is said to be ***convergent*** if there exists $L \in \mathbb{R}$ such that $\{a_n\}_{n=1}^{\infty}$ converges to L. If no such L exists we say that $\{a_n\}_{n=1}^{\infty}$ ***does not converge*** or simply ***diverges***.

Motivation: Intuitively, we expect that a sequence of real numbers $\{a_n\}_{n=1}^{\infty}$ converges to a number L if "outside any open interval containing L there are only finitely many $a_n's$". The statement inside " " can be translated into the following mathematical statement:

$$\forall\, \varepsilon > 0, \#\big\{n \in \mathbb{N} \,\big|\, a_n \notin (L-\varepsilon, L+\varepsilon)\big\} < \infty \,, \tag{9.1.1}$$

where $\#A$ denotes the number of points in the set A. One can easily show that the convergence of a sequence defined by (9.1.1) is equivalent to Definition 9.5.

In the definition above, we do not exclude the possibility that there are two different limits of a convergent sequence. In fact, this is never the case because of the following

Proposition 9.6

If $\{a_n\}_{n=1}^{\infty}$ is a sequence of real numbers, and $a_n \to a$ and $a_n \to b$ as $n \to \infty$, then $a = b$. (若收斂則極限唯一).

We will not prove this proposition and treat it as a fact.

• **Notation**: Since the limit of a convergent sequence is unique, we use $\lim\limits_{n\to\infty} a_n$ to denote this unique limit of a convergent sequence $\{a_n\}_{n=1}^{\infty}$.

Theorem 9.7

Let L be a real number, and $f : [1, \infty) \to \mathbb{R}$ be a function of a real variable such that $\lim_{x\to\infty} f(x) = L$. If $\{a_n\}_{n=1}^{\infty}$ is a sequence such that $f(n) = a_n$ for every positive integer n, then

$$\lim_{n\to\infty} a_n = L\,.$$

Example 9.8. The limit of the sequence $\{e_n\}_{n=1}^{\infty}$ defined by $e_n = \left(1 + \frac{1}{n}\right)^n$ is e.

When a sequence $\{a_n\}_{n=1}^{\infty}$ is given by evaluating a differentiable function $f : [1, \infty) \to \mathbb{R}$ on $\mathbb{N}$, sometimes we can use L'Hôspital's rule to find the limit of the sequence.

Example 9.9. The limit of the sequence $\{a_n\}_{n=1}^{\infty}$ defined by $a_n = \frac{n^2}{2^n - 1}$ is

$$\lim_{x\to\infty} \frac{x^2}{2^x - 1} = \lim_{x\to\infty} \frac{2x}{2^x \ln 2} = \lim_{x\to\infty} \frac{2}{2^x (\ln 2)^2} = 0\,.$$

There are cases that a sequence cannot be obtained by evaluating a function defined on $[1, \infty)$. In such cases, the limit of a sequence cannot be computed using L'Hôspital's rule and it requires more techniques to find the limit.

Example 9.10. The limit of the sequence $\{s_n\}_{n=1}^{\infty}$ defined by $s_n = \frac{n!}{n^{n+\frac{1}{2}} e^{-n}}$ is $\sqrt{2\pi}$; that is,

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n} n^n e^{-n}} = 1\,. \tag{9.1.2}$$

This is the Stirling formula. See Extra Exercise Problem Sets 6 from the previous semester.

Similar to Theorem 1.13, we have the following

Theorem 9.11

Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers such that $\lim_{n\to\infty} a_n = L$ and $\lim_{n\to\infty} b_n = K$. Then

1. $\lim_{n\to\infty} (a_n \pm b_n) = L \pm K$.
2. $\lim_{n\to\infty} (a_n b_n) = LK$. In particular, $\lim_{n\to\infty} (ca_n) = cL$ if c is a real number.
3. $\lim_{n\to\infty} \frac{a_n}{b_n} = \frac{L}{K}$ if $K \neq 0$.

Theorem 9.12: Squeeze Theorem

Let $\{a_n\}_{n=1}^\infty$, $\{b_n\}_{n=1}^\infty$ and $\{c_n\}_{n=1}^\infty$ be sequences of real numbers such that $a_n \leqslant c_n \leqslant b_n$ for all $n \geqslant N$. If $\lim\limits_{n\to\infty} a_n = \lim\limits_{n\to\infty} b_n = L$, then $\lim\limits_{n\to\infty} c_n = L$.

Theorem 9.13: Absolute Value Theorem

Let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers. If $\lim\limits_{n\to\infty} |a_n| = 0$, then $\lim\limits_{n\to\infty} a_n = 0$.

Proof. Let $\{b_n\}_{n=1}^\infty$ and $\{c_n\}_{n=1}^\infty$ be sequence of real numbers defined by $b_n = -|a_n|$ and $c_n = |a_n|$. Then $b_n \leqslant a_n \leqslant c_n$ for all $n \in \mathbb{N}$. Since $\lim\limits_{n\to\infty} |a_n| = 0$, Theorem 9.11 implies that $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} c_n = 0$ and the Squeeze Theorem further implies that $\lim\limits_{n\to\infty} a_n = 0$. □

Definition 9.14: Monotonicity of Sequences

A sequence $\{a_n\}_{n=1}^\infty \subseteq \mathbb{R}$ is said to be

1. (**monotone**) **increasing** if $a_{n+1} \geqslant a_n$ for all $n \in \mathbb{N}$;
2. (**monotone**) **decreasing** if $a_{n+1} \leqslant a_n$ for all $n \in \mathbb{N}$;
3. ***monotone*** if $\{a_n\}_{n=1}^\infty$ is an increasing sequence or a decreasing sequence.

Example 9.15. The sequence $\{s_n\}_{n=2}^\infty$ defined in Example 9.10 is a monotone decreasing sequence.

Definition 9.16: Boundedness of Sequences

Let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers.

1. $\{a_n\}_{n=1}^\infty$ is said to be ***bounded***（有界的）if there exists $M \in \mathbb{R}$ such that $|a_n| \leqslant M$ for all $n \in \mathbb{N}$.
2. $\{a_n\}_{n=1}^\infty$ is said to be ***bounded from above***（有上界）if there exists $B \in \mathbb{R}$, called an ***upper bound*** of the sequence, such that $a_n \leqslant B$ for all $n \in \mathbb{N}$. Such a number B is called an upper bound of the sequence.
3. $\{a_n\}_{n=1}^\infty$ is said to be ***bounded from below***（有下界）if there exists $A \in \mathbb{R}$, called a ***lower bound*** of the sequence, such that $A \leqslant a_n$ for all $n \in \mathbb{N}$. Such a number A is called a lower bound of the sequence.

Example 9.17. The sequence $\{a_n\}_{n=1}^{\infty}$ defined by $a_n = n$ is bounded from below by 0 but not bounded from above.

Proposition 9.18

A convergent sequence of real numbers is bounded（數列收斂必有界）.

Proof. Let $\{a_n\}_{n=1}^{\infty}$ be a convergent sequence with limit L. Then by the definition of limits of sequences, there exists $N > 0$ such that

$$a_n \in (L-1, L+1) \qquad \forall\, n \geqslant N.$$

Let $M = \max\big\{|a_1|, |a_2|, \cdots, |a_{N-1}|, |L|+1\big\}$. Then $|a_n| \leqslant M$ for all $n \in \mathbb{N}$. □

Remark 9.19. A bounded sequence might not be convergent. For example, let $\{a_n\}_{n=1}^{\infty}$ be defined by $a_n = 3 + (-1)^n$. Then

$$a_1 = a_3 = a_5 = \cdots = a_{2k-1} = \cdots = 2 \quad \text{and} \quad a_2 = a_4 = a_6 = \cdots = a_{2k} = \cdots = 4\,.$$

Therefore, the only possible limits are $\{2, 4\}$; however, by the fact that

$$\#\big\{n \in \mathbb{N} \,\big|\, a_n \notin (1,3)\big\} = \#\big\{n \in \mathbb{N} \,\big|\, a_n \notin (3,5)\big\} = \infty\,,$$

we find that 2 and 4 are not the limit of $\{a_n\}_{n=1}^{\infty}$. Therefore, $\{a_n\}_{n=1}^{\infty}$ does not converge.