Calculus MA1002-A Midterm 3

National Central University, May. 28, 2019

Problem 1. (20%) **True** or **False**（是非題）：每題兩分，答對得兩分，答錯倒扣兩分（倒扣至本大題零分為止）

In the following, R is always an open region in the plane, (a, b) is always a point in R, and $f : R \to \mathbb{R}$ is a function of two variables.

F 1. If $\lim_{t\to 0} f(a + t\cos\theta, b + t\sin\theta)$ exists for all $\theta \in \mathbb{R}$, then $\lim_{(x,y)\to(a,b)} f(x, y)$ exists.

T 2. If f is differentiable at (a, b), then f is continuous at (a, b).

F 3. If f_x and f_y both exist on R, then f is differentiable on R.

T 4. If f_x and f_y are continuous on R, then f is continuous on R.

T 5. If f_x and f_y both exist and are bounded on R, then f is continuous on R.

F 6. If $f_x(a, b)$ and $f_y(a, b)$ both exist, and $\boldsymbol{u}$ is a unit vector, then the directional derivative of f at (a, b) in the direction $\boldsymbol{u}$ is $\big(f_x(a, b), f_y(a, b)\big) \cdot \boldsymbol{u}$.

F 7. If the directional derivative of f at (a, b) exists in all directions, then f is continuous at (a, b).

F 8. If f_{xy} and f_{yx} both exist on R, then $f_{xy} = f_{yx}$ on R.

F 9. If f_x and f_y are continuous on R, then the level curve $f(x, y) = f(a, b)$ has a tangent line at (a, b).

T 10. If f_x and f_y are continuous on R and $(\nabla f)(a, b) \neq \mathbf{0}$, then the value of f at (a, b) increases most rapidly in the direction $\dfrac{(\nabla f)(a, b)}{\|(\nabla f)(a, b)\|}$.

Problem 2. Let R be an open region in the plane, $f : R \to \mathbb{R}$ be a function, and $(a, b) \in R$.

1. (5%) Define the differentiability of f at (a, b).
2. (5%) Define the directional derivative of f at (a, b) in direction $\mathbf{u}$, where $\boldsymbol{u} = (\cos\theta, \sin\theta)$ is a unit vector.

Problem 3. Assume that f is a continuous function of two variable satisfying that

$$\lim_{(x,y)\to(-1,1)} \frac{f(x, y) - 3x^2 + 2y^2}{\sqrt{(x + 1)^2 + (y - 1)^2}} = 0\,.$$

1. (10%) Find $f_x(-1, 1)$ and $f_y(-1, 1)$.
2. (5%) Prove or disprove that f is differentiable at $(-1, 1)$.

Solution. Note that since $\displaystyle\lim_{(x,y)\to(-1,1)} \frac{f(x,y)-3x^2+2y^2}{\sqrt{(x+1)^2+(y-1)^2}} = 0$, we must have

$$\lim_{(x,y)\to(-1,1)} \left[f(x,y)-3x^2+2y^2\right] = 0\,;$$

thus $\displaystyle\lim_{(x,y)\to(-1,1)} f(x,y) = 1$. Since f is continuous, $f(-1,1) = 1$.

For $(x,y) \neq (-1,1)$,

$$\begin{aligned}
\frac{f(x,y)-3x^2+2y^2}{\sqrt{(x+1)^2+(y-1)^2}} &= \frac{f(x,y)-3\big[(x+1)-1\big]^2+2\big[(y-1)+1\big]^2}{\sqrt{(x+1)^2+(y-1)^2}} \\
&= \frac{f(x,y)-3(x+1)^2+6(x+1)-3+2(y-1)^2+4(y-1)+2}{\sqrt{(x+1)^2+(y-1)^2}} \\
&= \frac{f(x,y)-f(-1,1)+6(x+1)+4(y-1)}{\sqrt{(x+1)^2+(y-1)^2}} + \frac{3(x+1)^2+2(y-1)^2}{\sqrt{(x+1)^2+(y-1)^2}}\,.
\end{aligned}$$

Since $\left|\dfrac{3(x+1)^2+2(y-1)^2}{\sqrt{(x+1)^2+(y-1)^2}}\right| \leqslant 3|x+1|+2|y-1|$, by Squeeze Theorem we find that

$$\lim_{(x,y)\to(-1,1)} \frac{3(x+1)^2+2(y-1)^2}{\sqrt{(x+1)^2+(y-1)^2}} = 0\,.$$

Therefore,

$$\lim_{(x,y)\to(-1,1)} \frac{f(x,y)-f(-1,1)+6(x+1)+4(y-1)}{\sqrt{(x+1)^2+(y-1)^2}} = 0$$

which implies that

$$\lim_{(x,y)\to(-1,1)} \frac{\big|f(x,y)-f(-1,1)+6(x+1)+4(y-1)\big|}{\sqrt{(x+1)^2+(y-1)^2}} = 0\,.$$

1. Note that the identity above implies that

$$\lim_{\substack{(x,y)\to(-1,1)\\ y=1}} \frac{\big|f(x,y)-f(-1,1)+6(x+1)+4(y-1)\big|}{\sqrt{(x+1)^2+(y-1)^2}} = 0\,.$$

Therefore,

$$\begin{aligned}
0 &= \lim_{\substack{(x,y)\to(-1,1)\\ y=1}} \left|\frac{\big|f(x,y)-f(-1,1)+6(x+1)+4(y-1)\big|}{\sqrt{(x+1)^2+(y-1)^2}}\right| \\
&= \lim_{x\to-1} \left|\frac{f(x,1)-f(-1,1)+6(x+1)}{x+1}\right| = \lim_{x\to-1}\left|\frac{f(x,1)-f(-1,1)}{x-(-1)}+6\right|;
\end{aligned}$$

thus

$$\underline{f_x(-1,1) = \lim_{x\to-1} \frac{f(x,1)-f(-1,1)}{x-(-1)} = -6\,.}$$

Similarly, $\underline{f_y(-1,1) = -4}$.

2. In the computations above, we conclude that

$$\lim_{(x,y)\to(-1,1)} \left|\frac{\big|f(x,y)-f(-1,1)-f_x(-1,1)(x+1)-f_y(-1,1)(y-1)\big|}{\sqrt{(x+1)^2+(y-1)^2}}\right| = 0\,.$$

By definition, $\underline{f \text{ is differentiable at } (-1,1)}$. □

Problem 4. (10%) Let $f, g : \mathbb{R}^2 \to \mathbb{R}$ be defined by

$$f(x,y) = \begin{cases} \dfrac{x^2(x+y)}{x^2+y^4} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

Find the directional derivative of f at $(0,0)$ in the direction along which the value of the function f at $(0,0)$ decreases most rapidly.

Solution. Let $\boldsymbol{u}$ be the direction along which the value of the function f at $(0,0)$ decreases most rapidly. Then

$$(D_{\boldsymbol{u}}f)(0,0) = \min\left\{(D_{\boldsymbol{v}}f)(0,0) \,\middle|\, \|\boldsymbol{v}\| = 1\right\}.$$

Let $\boldsymbol{v} = (\cos\theta, \sin\theta)$. Then

$$\begin{aligned}(D_{\boldsymbol{v}}f)(0,0) &= \lim_{t\to 0} \frac{f(t\cos\theta, t\sin\theta) - f(0,0)}{t} = \lim_{t\to 0} \frac{t^3\cos^2\theta(\cos\theta+\sin\theta)}{t^3(\cos^2\theta + t^2\sin^4\theta)} \\ &= \lim_{t\to 0} \frac{\cos^2\theta(\cos\theta+\sin\theta)}{\cos^2\theta + t^2\sin^4\theta}\,.\end{aligned}$$

If $\cos\theta = 0$, then $(D_{\boldsymbol{v}}f)(0,0) = 0$. If $\cos\theta \neq 0$, then $(D_{\boldsymbol{v}}f)(0,0) = \cos\theta + \sin\theta$. Therefore,

$$(D_{\boldsymbol{v}}f)(0,0) = \begin{cases} 0 & \text{if } \cos\theta = 0\,, \\ \cos\theta + \sin\theta & \text{if } \cos\theta \neq 0\,. \end{cases}$$

Since $\min\left\{\cos\theta + \sin\theta \,\middle|\, \theta \in [0, 2\pi)\right\} = -\sqrt{2}$ $\left(\text{attained at } \theta = \dfrac{3\pi}{4}\right)$; thus $\underline{(D_{\boldsymbol{u}}f)(0,0) = -\sqrt{2}}$. □

Problem 5. (15%) Find the second Taylor polynomial of the function $f(x,y) = \arctan\dfrac{y+1}{x+1}$ at $(0,0)$.

Solution. First, $f(0,0) = \arctan 1 = \dfrac{\pi}{4}$. By the chain rule, for $x \neq -1$,

$$\begin{aligned} f_x(x,y) &= \frac{\frac{\partial}{\partial x}\frac{y+1}{x+1}}{1+(\frac{y+1}{x+1})^2} = \frac{-\frac{y+1}{(x+1)^2}}{1+(\frac{y+1}{x+1})^2} = -\frac{y+1}{(x+1)^2+(y+1)^2}\,, \\ f_y(x,y) &= \frac{\frac{\partial}{\partial y}\frac{y+1}{x+1}}{1+(\frac{y+1}{x+1})^2} = \frac{\frac{1}{x+1}}{1+(\frac{y+1}{x+1})^2} = \frac{x+1}{(x+1)^2+(y+1)^2}\,, \end{aligned}$$

and

$$\begin{aligned} f_{xx}(x,y) &= \frac{2(x+1)(y+1)}{\big[(x+1)^2+(y+1)^2\big]^2}\,, \quad f_{yy}(x,y) = \frac{-2(x+1)(y+1)}{\big[(x+1)^2+(y+1)^2\big]^2} \\ f_{xy}(x,y) &= -\frac{(x+1)^2+(y+1)^2-2(y+1)^2}{\big[(x+1)^2+(y+1)^2\big]^2} = \frac{(y+1)^2-(x+1)^2}{\big[(x+1)^2+(y+1)^2\big]^2}\,. \end{aligned}$$

Therefore, the second Taylor's polynomial of f is

$$\begin{aligned} &f(0,0) + f_x(0,0)x + f_y(0,0)y + \frac{1}{2}\Big[f_{xx}(0,0)x^2 + 2f_{xy}(0,0)xy + f_{yy}(0,0)y^2\Big] \\ &= \frac{\pi}{4} - \frac{1}{2}x + \frac{1}{2}y + \frac{1}{2}\big(\frac{1}{2}x^2 - \frac{1}{2}y^2\big) = \underline{\frac{\pi}{4} - \frac{1}{2}x + \frac{1}{2}y + \frac{1}{4}\big(x^2-y^2\big)}\,. \end{aligned}$$

□

Problem 6. (10%) Find all relative extrema and saddle points of $f(x,y) = (x^2+y^2)e^{y^2-x^2}$ using the second derivative test. When a relative extremum is found, determine if it is a relative maximum or a relative minimum.

Solution. We first compute the first and second partial derivatives of f and find that

$$\begin{aligned} f_x(x,y) &= 2xe^{y^2-x^2} + (x^2+y^2)(-2x)e^{y^2-x^2} = 2x(1-x^2-y^2)e^{y^2-x^2}\,, \\ f_y(x,y) &= 2ye^{y^2-x^2} + (x^2+y^2)(2y)e^{y^2-x^2} = 2y(1+x^2+y^2)e^{y^2-x^2}\,, \\ f_{xx}(x,y) &= \big[2-6x^2-2y^2-4x^2(1-x^2-y^2)\big]e^{y^2-x^2}\,, \\ f_{xy}(x,y) &= \big[2x(-2y)+4xy(1-x^2-y^2)\big]e^{y^2-x^2}\,, \\ f_{yy}(x,y) &= \big[2+2x^2+6y^2+4y^2(1+x^2+y^2)\big]e^{y^2-x^2}\,. \end{aligned}$$

Therefore, critical points of f are $(0,0)$, $(1,0)$ and $(-1,0)$.

1. Since $f_{xx}(0,0) = f_{yy}(0,0) = 2$, $f_{xy}(0,0) = 0$, we find that $f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = 4 > 0$; thus the fact that $f_{xx}(0,0) > 0$ implies that $\underline{f(0,0) \text{ is a relative minimum of } f}$.

2. Since $f_{xx}(1,0) = -4e^{-1}$, $f_{yy}(1,0) = 4e^{-1}$ and $f_{xy}(1,0) = 0$, we find that $f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = -16e^{-2} < 0$; thus $\underline{(1,0) \text{ is a saddle point of } f}$.

3. Since $f_{xx}(-1,0) = -4e^{-1}$, $f_{yy}(-1,0) = 4e^{-1}$ and $f_{xy}(-1,0) = 0$, we find that $f_{xx}(0,0)f_{yy}(0,0) - f_{xy}(0,0)^2 = -16e^{-2} < 0$; thus $\underline{(-1,0) \text{ is a saddle point of } f}$. □

Problem 7. (20%) Let R be the solid in the space given by

$$\big\{(x,y,z)\,\big|\,1 \leqslant z \leqslant \sqrt{4-x^2-y^2}\big\}\,.$$

Find the extreme value of function $w = f(x,y,z) = xyz$ on R.

Solution. Let $g(x,y,z) = x^2+y^2+z^2-4$, and $h(x,y,z) = z-1$. Then

$$\begin{aligned} (\nabla f)(x,y,z) &= (yz, xz, xy)\,, \\ (\nabla g)(x,y,z) &= (2x, 2y, 2z)\,, \\ (\nabla h)(x,y,z) &= (0,0,1)\,. \end{aligned}$$

If $(\nabla f)(x,y,z) = 0$, then $xy = yz = zx = 0$ which implies that at least two of x,y,z are zero. In this case, $f(x,y,z) = 0$.

Now we consider the extreme value of f on the boundary of R. Suppose that the extreme value of f occurs at (x_0,y_0,z_0). Note that the boundary of R consists of three pieces: $g = 0$, $h = 0$ and $g = h = 0$.

1. $g(x_0,y_0,z_0) = 0$: Since $(\nabla g)(x_0,y_0,z_0) \neq \mathbf{0}$, Lagrange Multiplier Theorem implies that there exists $\lambda \in \mathbb{R}$ such that

$$(y_0z_0, x_0z_0, x_0y_0) = \lambda(2x_0, 2y_0, 2z_0)\,.$$

Therefore, (x_0, y_0, z_0, λ) satisfies

$$y_0 z_0 = 2\lambda x_0\,, \tag{0.1a}$$
$$x_0 z_0 = 2\lambda y_0\,, \tag{0.1b}$$
$$x_0 y_0 = 2\lambda z_0\,, \tag{0.1c}$$
$$x_0^2 + y_0^2 + z_0^2 = 4\,. \tag{0.1d}$$

If one of x_0, y_0, z_0 is zero, then $f(x_0, y_0, z_0) = 0$; thus we assume that $x_0 y_0 z_0 \neq 0$. Then $\lambda \neq 0$ and the product of (0.1a,b,c) shows that $x_0 y_0 z_0 = 8\lambda^3$. Therefore,

$$x_0 = \frac{4\lambda^2}{x_0}\,, \quad y_0 = \frac{4\lambda^2}{y_0}\,, \quad z_0 = \frac{4\lambda^2}{z_0}\,.$$

which implies that (x_0, y_0, z_0) is

$$(\pm 2\lambda, \pm 2\lambda, 2\lambda)\,, \quad (\pm 2\lambda, \mp 2\lambda, -2\lambda)\,, \quad (\pm 2\lambda, 2\lambda, \pm 2\lambda)\,,$$
$$(\pm 2\lambda, -2\lambda, \mp 2\lambda)\,, \quad (2\lambda, \pm 2\lambda, \pm 2\lambda)\,, \quad (-2\lambda, \pm 2\lambda, \mp 2\lambda)\,.$$

In either cases, (0.1d) implies that $12\lambda^2 = 4$; thus $\lambda = \pm\frac{1}{\sqrt{3}}$. Since $z_0 \geqslant 1$, we conclude that

$$(x_0, y_0, z_0) = \Big(\pm\frac{2}{\sqrt{3}}, \pm\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\Big) \quad \text{or} \quad \Big(\pm\frac{2}{\sqrt{3}}, \mp\frac{2}{\sqrt{3}}, \frac{2}{\sqrt{3}}\Big)\,.$$

In this case, $f(x_0, y_0, z_0) = \pm\frac{8}{3\sqrt{3}}$.

2. $h(x_0, y_0, z_0) = 0$: Since $(\nabla h)(x_0, y_0, z_0) \neq \mathbf{0}$, Lagrange Multiplier Theorem implies that there exists $\lambda \in \mathbb{R}$ such that

$$(y_0 z_0, x_0 z_0, x_0 y_0) = \lambda(0, 0, 1) \quad \text{and} \quad z_0 = 1\,.$$

Therefore, $(x_0, y_0, z_0) = (0, 0, 1)$ which is impossible $f(x_0, y_0, z_0) = 0$.

3. $g(x_0, y_0, z_0) = h(x_0, y_0, z_0) = 0$: Since

$$(\nabla g)(x_0, y_0, z_0) \times (\nabla h)(x_0, y_0, z_0) = (2x_0, 2y_0, 2z_0) \times (1, 1, 1) = 2(y_0 - z_0, z_0 - x_0, x_0 - y_0)\,,$$

$(\nabla g)(x_0, y_0, z_0) \times (\nabla h)(x_0, y_0, z_0) = \mathbf{0}$ if and only if $x_0 = y_0 = z_0$. Since $h(x_0, y_0, z_0) = 0$ implies that $z_0 = 1$, and $g(1, 1, 1) \neq 0$, we find that $(\nabla g)(x_0, y_0, z_0) \times (\nabla h)(x_0, y_0, z_0) = \mathbf{0}$. Therefore, Lagrange Multiplier Theorem implies that there exist $\lambda\mu \in \mathbb{R}$ such that

$$(y_0 z_0, x_0 z_0, x_0 y_0) = \lambda(2x_0, 2y_0, 2z_0) + \mu(0, 0, 1)\,.$$

Therefore, $(x_0, y_0, z_0, \lambda, \mu)$ satisfies

$$y_0 z_0 = 2\lambda x_0\,, \tag{0.2a}$$
$$x_0 z_0 = 2\lambda y_0\,, \tag{0.2b}$$
$$x_0 y_0 = 2\lambda z_0 + \mu\,, \tag{0.2c}$$
$$x_0^2 + y_0^2 + z_0^2 = 4\,, \tag{0.2d}$$
$$z_0 = 1\,. \tag{0.2e}$$

By (0.2a,b,e), we find that $x_0 = 2\lambda y_0 = 4\lambda^2 x_0$; thus $x_0 = 0$ or $4\lambda^2 = 1$.

(a) If $x_0 = 0$, then $f(x_0, y_0, z_0) = 0$.

(b) If $x_0 \neq 0$, then $\lambda = \pm\frac{1}{2}$.

(i) $\lambda = \frac{1}{2}$: (0.2a,e) implies that $y_0 = x_0$; thus (0.2) implies that $(x_0, y_0, z_0) = \big(\sqrt{\frac{3}{2}}, \sqrt{\frac{3}{2}}, 1\big)$. In this case, $f(x_0, y_0, z_0) = \frac{3}{2}$.

(ii) $\lambda = -\frac{1}{2}$: (0.2a,e) implies that $y_0 = -x_0$; thus (0.2) implies that $(x_0, y_0, z_0) = \big(\pm\sqrt{\frac{3}{2}}, \mp\sqrt{\frac{3}{2}}, 1\big)$. In this case, $f(x_0, y_0, z_0) = -\frac{3}{2}$.

Comparing the values of all possible extreme points (x_0, y_0, z_0), we find that the maximum of f on R is $\frac{8}{3\sqrt{3}}$, and the minimum of f on R is $-\frac{8}{3\sqrt{3}}$. □