Calculus MA1002-A Midterm 2

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Problem 1. (10%) Suppose that the limit $\lim_{n\to\infty} n^\alpha r^n C_n^{3n}$ exists and is non-zero. Find α, r and the limit.

Solution. Recall the Stirling formula:

$$\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n} n^n e^{-n}} = 1 .$$

Therefore,

$$\lim_{n\to\infty} \frac{\sqrt{2\pi(3n)}(3n)^{3n}e^{-3n}}{(3n)!} = \lim_{n\to\infty} \frac{(2n)!}{\sqrt{2\pi(2n)}(2n)^{2n}e^{-2n}} = 1 .$$

By definition, $C_n^{3n} = \dfrac{(3n)!}{n!(2n)!}$; thus if the limit $\lim_{n\to\infty} n^\alpha r^n C_n^{3n}$ exists,

$$\begin{aligned}
&\lim_{n\to\infty} n^\alpha r^n C_n^{3n} \\
&= \Big(\lim_{n\to\infty} n^\alpha r^n C_n^{3n}\Big)\Big(\lim_{n\to\infty} \frac{n!}{\sqrt{2\pi n} n^n e^{-n}}\Big)\Big(\lim_{n\to\infty} \frac{(2n)!}{\sqrt{2\pi(2n)}(2n)^{2n}e^{-2n}}\Big)\Big(\lim_{n\to\infty} \frac{\sqrt{2\pi(3n)}(3n)^{3n}e^{-3n}}{(3n)!}\Big) \\
&= \lim_{n\to\infty} \Big(n^\alpha r^n C_n^{3n} \frac{n!(2n)!}{(3n)!} \frac{\sqrt{2\pi(3n)}(3n)^{3n}e^{-3n}}{\sqrt{2\pi n} n^n e^{-n}\sqrt{2\pi(2n)}(2n)^{2n}e^{-2n}}\Big) \\
&= \lim_{n\to\infty} \Big(n^\alpha r^n \frac{\sqrt{2\pi(3n)}(3n)^{3n}e^{-3n}}{\sqrt{2\pi n} n^n e^{-n}\sqrt{2\pi(2n)}(2n)^{2n}e^{-2n}}\Big) = \lim_{n\to\infty} \Big(n^\alpha r^n \frac{\sqrt{2\pi(3n)}3^{3n}}{\sqrt{2\pi n}\sqrt{2\pi(2n)}2^{2n}}\Big) \\
&= \sqrt{\frac{3}{4\pi}} \lim_{n\to\infty} \Big[n^{\alpha-\frac{1}{2}}\Big(\frac{27r}{4}\Big)^n\Big] .
\end{aligned}$$

Since the limit $\lim_{n\to\infty} n^\beta s^n$ does not exist unless $\beta = 0$ and $s = 1$, we conclude that if the limit $\lim_{n\to\infty} n^\alpha r^n C_n^{3n}$ exists, $\underline{\alpha = \frac{1}{2}}$ and $\underline{r = \frac{4}{27}}$, and in such a case $\underline{\lim_{n\to\infty} n^\alpha r^n C_n^{3n} = \sqrt{\frac{3}{4\pi}}}$. □

Problem 2. (15%) Find all value $p \in \mathbb{R}$ such that $\sum\limits_{k=2}^\infty \Big[\exp\Big(\frac{1}{k(\ln k)^p}\Big) - 1\Big]$ converges. Note that you need to provide the reason for the convergence or divergence of the power series for each p.

Solution. Let $a_n = \exp\Big(\dfrac{1}{n(\ln n)^p}\Big) - 1$ and $b_n = \dfrac{1}{n(\ln n)^p}$. Then $a_n, b_n \geqslant 0$ for all $n \geqslant 2$. Moreover, $\lim_{n\to\infty} b_n = 0$; thus by the fact that $a_n = \exp(b_n) - 1$ and $\lim_{x\to 0} \dfrac{e^x - 1}{x} = 1$, we find that $\lim_{n\to\infty} \dfrac{a_n}{b_n} = 1$. Therefore, the limit comparison test implies that $\sum\limits_{n=2}^\infty a_n$ converges if and only if $\sum\limits_{n=2}^\infty b_n$ converges.

If $p > 0$, then the function $y = \dfrac{1}{x(\ln x)^p}$ is decreasing on $[2, \infty)$; thus the integral test implies that $\sum\limits_{n=2}^\infty b_n$ converges if and only if $\displaystyle\int_2^\infty \frac{1}{x(\ln x)^p}\,dx$ converges. A substitution of variable shows that

$$\int_2^\infty \frac{1}{x(\ln x)^p}\,dx \overset{(x=e^u)}{=} \int_{\ln 2}^\infty \frac{1}{u^p e^u} e^u\,du = \int_{\ln 2}^\infty u^{-p}\,du$$

which converges if and only if $p > 1$. Therefore, if $p > 0$, then $\sum\limits_{n=2}^{\infty} b_n$ converges if and only if $p > 1$.

For $p \leqslant 0$, note that $0 < \dfrac{1}{k} \leqslant \dfrac{1}{k(\ln k)^p}$ for all $k \geqslant 3$; thus by the fact that $\sum\limits_{k=1}^{\infty} \dfrac{1}{k}$ diverges, the comparison test implies that $\sum\limits_{k=2}^{\infty} \dfrac{1}{k(\ln k)^p}$ diverges for $p \leqslant 0$.

Combining the discussion above, we conclude that

$$\sum_{k=2}^{\infty} \left[\exp\Big(\frac{1}{k(\ln k)^p}\Big) - 1 \right] \text{ converges if and only if } p > 1. \qquad \square$$

Problem 3. (15%) Show that $\sum\limits_{k=1}^{\infty} \dfrac{\sin(kx)}{k}$ converges for all $x \in \mathbb{R}$.

Proof. First we note that by the periodicity of the sine function, it suffices to show that $\sum\limits_{k=1}^{\infty} \dfrac{\sin(kx)}{k}$ converges for all $0 \leqslant x \leqslant 2\pi$. Moreover, if $x = 0$ or $x = 2\pi$, the sum is clearly 0; thus we only need to show that $\sum\limits_{k=1}^{\infty} \dfrac{\sin(kx)}{k}$ converges for all $x \in (0, 2\pi)$.

For $x \in (0, 2\pi)$, $\sin\dfrac{x}{2} \neq 0$; thus the fact that

$$\begin{aligned}
2\sin\frac{x}{2}\sum_{k=1}^{n}\sin(kx) &= \sum_{k=1}^{n}\Big[\cos\big(kx - \frac{x}{2}\big) - \cos\big(kx + \frac{x}{2}\big)\Big] = \sum_{n=1}^{\infty}\Big[\cos\big(k - \frac{1}{2}\big)x - \cos\big(k + \frac{1}{2}\big)x\Big] \\
&= \Big(\cos\frac{x}{2} - \cos\frac{3x}{2}\Big) + \Big(\cos\frac{3x}{2} - \cos\frac{5x}{2}\Big) + \cdots + \Big(\cos\big(n - \frac{1}{2}\big)x - \cos\big(n + \frac{1}{2}\big)x\Big) \\
&= \cos\frac{x}{2} - \cos\frac{(2n+1)x}{2},
\end{aligned}$$

we have $\sum\limits_{k=1}^{n} \sin(kx) = \dfrac{\cos\frac{x}{2} - \cos\frac{(2n+1)x}{2}}{2\sin\frac{x}{2}}$. Therefore, for each $n \in \mathbb{N}$ and $x \in (0, 2\pi)$,

$$\Big|\sum_{k=1}^{n}\sin(kx)\Big| \leqslant \frac{1}{\sin\frac{x}{2}} < \infty\,.$$

Therefore, by the Abel (or Dirichlet) test, $\sum\limits_{k=1}^{\infty} \dfrac{\sin(kx)}{k}$ converges for all $x \in (0, 2\pi)$. $\square$

Problem 4. (15%) Find the radius of convergence and the interval of convergence of the power series

$$\sum_{k=2}^{\infty} \frac{(-1)^k x^{2^k}}{2^k \ln k}\,.$$

Solution. Let $a_n = \dfrac{(-1)^n x^{2^n}}{2^n \ln n}$. Then

$$\Big|\frac{a_{n+1}}{a_n}\Big| = \frac{\frac{x^{2^{n+1}}}{2^{n+1}\ln(n+1)}}{\frac{x^{2^n}}{2^n \ln n}} = \frac{\ln n}{2\ln(n+1)} x^{2^{n+1}-2^n} = \frac{\ln n}{2\ln(n+1)} x^{2^n}\,.$$

If $|x| > 1$, then $\lim\limits_{n\to\infty} \Big|\dfrac{a_{n+1}}{a_n}\Big| = \infty$. On the other hand, if $|x| \leqslant 1$, then

$$\lim_{n\to\infty}\Big|\frac{a_{n+1}}{a_n}\Big| = \begin{cases} 0 & \text{if } |x| < 1\,, \\ \dfrac{1}{2} & \text{if } |x| = 1\,. \end{cases}$$

Therefore, the ratio test shows that $\sum\limits_{k=2}^{\infty} \frac{(-1)^k x^{2^k}}{2^k \ln k}$ converges if and only if $|x| \leqslant 1$; thus

1. the radius of convergence is 1;   2. the interval of convergence is $[-1, 1]$. $\square$

Problem 5. Suppose that $x(t)$ is a function of t satisfying the following equations

$$x''(t) + x(t) = 0\,, \quad x(0) = 1\,, \quad x'(0) = 1\,,$$

where $'$ denotes the derivatives with respect to t.

1. (5%) Assume that the function $x(t)$ can be written as a power series (on a certain interval), that is, $x(t) = \sum\limits_{k=0}^{\infty} a_k t^k$. Show that $(k+2)(k+1)a_{k+2} + a_k = 0$ for all $k \geqslant 0$.

2. (10%) Show that the 4-th Maclaurin polynomial of $\sin t + \cos t$ agrees with the 4-th Maclaurin polynomial of $x(t)$.

Proof. 1. Suppose that $x(t) = \sum\limits_{k=0}^{\infty} a_k t^k$ has radius of convergence R and is a solution to the equation. Since

$$x''(t) = \sum_{k=2}^{\infty} k(k-1)a_k t^{k-2} = \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} t^k \qquad \forall\, |t| < R\,,$$

we find that for $t \in (-R, R)$,

$$0 = \sum_{k=0}^{\infty} (k+2)(k+1)a_{k+2} t^k + \sum_{k=0}^{\infty} a_k t^k = \sum_{k=0}^{\infty} \big[(k+2)(k+1)a_{k+2} + a_k\big] t^k\,.$$

Therefore, $(k+2)(k+1)a_{k+2} + a_k = 0$ for all $k \geqslant 0$.

2. Since $x(0) = x'(0) = 1$, we find that $a_0 = a_1 = 1$. Therefore,

$$a_2 = \frac{-a_0}{2 \cdot 1} = -\frac{1}{2}\,, \quad a_3 = \frac{-a_1}{3 \cdot 2} = -\frac{1}{6}\,, \quad a_4 = \frac{-a_2}{4 \cdot 3} = \frac{1}{24}\,;$$

thus the 4-th Maclaurin polynomial of x is $1 + t - \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24}$. On the other hand, the Maclaurin series of the function $y = \sin t + \cos t$ is the sum of the Maclaurin series of the sine function and the Maclaurin series of the cosine function; thus the Maclaurin series of $\sin t + \cos t$ is

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)!} t^{2k+1} + \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} t^{2k}$$

which shows the 4-th Maclaurin polynomial of $\sin t + \cos t$ is

$$t - \frac{t^3}{3!} + 1 - \frac{t^2}{2!} + \frac{t^4}{4!} = 1 - t - \frac{t^2}{2} - \frac{t^3}{6} + \frac{t^4}{24}\,.$$

Therefore, the 4-th Maclaurin polynomial of $\sin t + \cos t$ agrees with the 4-th Maclaurin polynomial of $x(t)$. $\square$

Problem 6. Complete the following.

1. (5%) State the Taylor Theorem (for functions of one variable).

2. (10%) Use the Taylor Theorem to show that

$$\arctan x \leqslant \sum_{k=0}^{2n} (-1)^k \frac{x^{2k+1}}{2k+1} \qquad \forall\, x > 0\,.$$

Proof of 2. Let $f(x) = \dfrac{1}{1+x}$. Then $f^{(k)}(x) = (-1)^k k!(1+x)^{-(k+1)}$. Therefore, Taylor's Theorem implies that for each $x \in \mathbb{R}$, there exists ξ between x and 0 such that

$$f(x) = \sum_{k=0}^{2n} \frac{f^{(k)}(0)}{k!} x^k + \frac{f^{(2n+1)}(\xi)}{(2n+1)!} x^{2n+1} = \sum_{k=0}^{2n} (-1)^k x^k - (1+\xi)^{-2n-2} x^{2n+1}\,.$$

Therefore, if $x > 0$,

$$\frac{1}{1+x} = \sum_{k=0}^{2n} (-1)^k x^k - (1+\xi)^{-2n-2} x^{2n+1} \leqslant \sum_{k=0}^{2n} (-1)^k x^k\,.$$

In particular, for all $x \in \mathbb{R}$,

$$\frac{1}{1+x^2} \leqslant \sum_{k=0}^{2n} (-1)^k x^{2k}\,;$$

thus if $x > 0$,

$$\arctan x = \int_0^x \frac{1}{1+t^2}\,dt \leqslant \int_0^x \sum_{k=0}^{2n} (-1)^k t^{2k}\,dt = \sum_{k=0}^{2n} (-1)^k \int_0^x t^{2k}\,dt = \sum_{k=0}^{2n} \frac{(-1)^k}{2k+1} x^{2k+1}\,.$$

□

Problem 7. (15%) Find n such that

$$\Big| e - \sum_{k=0}^{n} \frac{1}{k!} \Big| < 5 \times 10^{-6}\,.$$

Explain your answer.

Solution. By Taylor's Theorem, for each $x \in \mathbb{R}$ there exists ξ between 0 and x such that

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!} + \frac{e^\xi}{(n+1)!} x^{n+1}\,.$$

In particular, there exists $\xi \in (0,1)$ such that $e = \sum\limits_{k=0}^{n} \dfrac{1}{k!} + \dfrac{e^\xi}{(n+1)!}$. Therefore,

$$\Big| e - \sum_{k=0}^{n} \frac{1}{k!} \Big| \leqslant \frac{e}{(n+1)!} \leqslant \frac{3}{(n+1)!}\,.$$

Choosing $n = 15$, we find that

$$\Big| e - \sum_{k=0}^{n} \frac{1}{k!} \Big| \leqslant \frac{3}{15!} \leqslant \frac{3}{10 \cdot 11 \cdot 12 \cdot 13 \cdot 14 \cdot 15} \leqslant 3 \cdot 10^{-6} < 5 \times 10^{-6}\,.$$

In fact, the desired inequality holds as long as $n \geqslant 10$. □