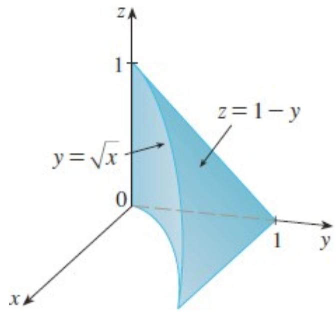
Calculus MA1002-A Final Exam

National Central University, Jun. 20, 2019

Problem 1. (10%) Rewrite the iterated integral $\int_0^1 \Big[\int_{\sqrt{x}}^1 \Big(\int_0^{1-y} f(x,y,z)\,dz\Big)dy\Big]dx$ in the order $dxdydz$.

Solution. Note that the region of integration Q is shown in the following figure



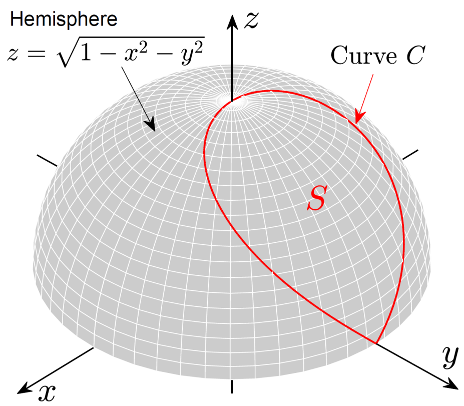
Let R be the projection of Q along the x-axis onto the yz-plane. Then R is the triangle given by

$$R = \big\{(y,z)\,\big|\,0 \leqslant z \leqslant 1, 0 \leqslant y \leqslant 1-z\big\}\,.$$

Therefore,

$$\int_0^1 \Big[\int_{\sqrt{x}}^1 \Big(\int_0^{1-y} f(x,y,z)\,dz\Big)dy\Big]dx = \int_0^1 \Big[\int_0^{1-z} \Big(\int_0^{y^2} f(x,y,z)\,dx\Big)\,dy\Big]\,dz\,. \qquad \square$$

Problem 2. Let S be the subset of the upper hemisphere $z = \sqrt{1-x^2-y^2}$ enclosed by the curve C shown in the figure below



where each point of C corresponds to some point $(\cos t \sin t, \sin^2 t, \cos t)$ with $t \in \big[-\frac{\pi}{2}, \frac{\pi}{2}\big]$. Find the surface of S via the following steps:

1. (10%) Let R be the region obtained by projecting S onto the xy-plane along the z-axis. Suppose that R can be expressed as $R = \big\{(x,y)\,\big|\,c \leqslant y \leqslant d, g_1(y) \leqslant x \leqslant g_2(y)\big\}$. Find c, d and g_1, g_2.

2. (10%) In polar coordinate (with $(0,0)$ as the pole and x-axis as the polar axis), the region R in the xy-plane corresponds to the region R' in the $r\theta$-plane

$$R' = \{(r,\theta) \,|\, a \leqslant \theta \leqslant b, h_1(\theta) \leqslant r \leqslant h_2(\theta)\}.$$

Find a, b and h_1, h_2.

3. (5%) The surface area of S can be computed by $\iint\limits_R f(x,y)\,dA$. Find $f(x,y)$.

4. (15%) Use polar coordinate as the change of variables to compute $\iint\limits_R f(x,y)\,dA$.

5. (15%) Let D be the solid region above R and below S; that is,

$$D = \{(x,y,z) \,|\, (x,y) \in R, 0 \leqslant z \leqslant \sqrt{1-x^2-y^2}\}.$$

Find the volume of D.

Solution. 1. Let (x,y) be a boundary point of R. The $(x,y) = (\cos t \sin t, \sin^2 t)$ for some $t \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$; thus

$$x^2 + y^2 = \cos^2 t \sin^2 t + \sin^4 t = (\cos^2 t + \sin^2 t)\sin^2 t = \sin^2 t = y.$$

Therefore, the boundary of R consists of points (x,y) satisfying $x^2 + y^2 = y$ which shows that R is a disk centered at $\left(0, \frac{1}{2}\right)$ with radius $\frac{1}{2}$. Therefore,

$$R = \{(x,y) \,|\, 0 \leqslant y \leqslant 1, -\sqrt{y-y^2} \leqslant x \leqslant \sqrt{y-y^2}\}.$$

2. By the fact that the boundary of R' maps to the boundary of R under the change of variables $x = r\cos\theta$ and $y = r\sin\theta$, we find that if (r,θ) is a boundary point of R', then (r,θ) satisfies

$$r^2 = r\sin\theta.$$

Therefore, the boundary of R' consists of points (r,θ) satisfying $r = \sin\theta$ or $r = 0$ in the $r\theta$-plane. Since R locates on the upper half plane, $0 \leqslant \theta \leqslant \pi$, and the center of the disk R corresponds to point $\left(\frac{1}{2}, \frac{\pi}{2}\right)$ in the $r\theta$-plane, we conclude that

$$R' = \{(r,\theta) \,|\, 0 \leqslant \theta \leqslant \pi, 0 \leqslant r \leqslant \sin\theta\}.$$

3. Since $\dfrac{\partial z}{\partial x} = \dfrac{-x}{\sqrt{1-x^2-y^2}}$ and $\dfrac{\partial z}{\partial y} = \dfrac{-y}{\sqrt{1-x^2-y^2}}$, the surface area of S is given by

$$\iint\limits_R \sqrt{1 + \left(\frac{\partial z}{\partial x}\right)^2 + \left(\frac{\partial z}{\partial x}\right)^2}\,dA = \iint\limits_R \frac{1}{\sqrt{1-x^2-y^2}}\,dA.$$

4. Using the polar coordinate as the change of variables,

$$\begin{aligned}
\int_0^\pi \left(\int_0^{\sin\theta} \frac{1}{\sqrt{1-r^2}}\,r\,dr\right)d\theta &= \int_0^\pi \left[\left(-\sqrt{1-r^2}\right)\Big|_{r=0}^{r=\sin\theta}\right]d\theta = \int_0^\pi \left(1 - |\cos\theta|\right)d\theta \\
&= \pi - 2\int_0^{\frac{\pi}{2}} \cos\theta\,d\theta = \pi - 2\left(\sin\theta\Big|_{\theta=0}^{\theta=\frac{\pi}{2}}\right) = \pi - 2.
\end{aligned}$$

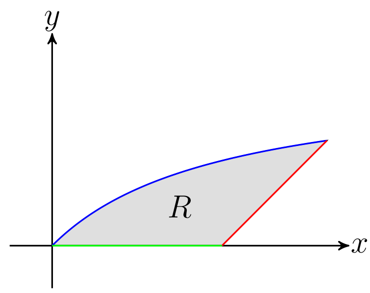
5. Using the cylindrical coordinate, the volume of D is given by

$$\iint_{R'} \Big(\int_0^{\sqrt{1-x^2-y^2}} dz \Big) r d(r,\theta) = \int_0^{\pi} \Big(\int_0^{\sin\theta} r\sqrt{1-r^2}\, dr \Big) d\theta = \int_0^{\pi} \Big[\Big(-\frac{1}{3}(1-r^2)^{\frac{3}{2}}\Big)\Big|_{r=0}^{r=\sin\theta} \Big] d\theta$$

$$= \frac{1}{3}\int_0^{\pi} \big(1 - |\cos\theta|^3\big)\, d\theta = \frac{1}{3}\Big(\pi - 2\int_0^{\frac{\pi}{2}} \cos^3\theta\, d\theta\Big) = \frac{1}{3}\Big(\pi - 2\int_0^{\frac{\pi}{2}} \frac{\cos 3\theta + 3\cos\theta}{4}\, d\theta\Big)$$

$$= \frac{\pi}{3} - \frac{1}{6}\Big[\Big(\frac{\sin 3\theta}{3} + 3\sin\theta\Big)\Big|_{\theta=0}^{\theta=\frac{\pi}{2}}\Big] = \frac{\pi}{3} - \frac{1}{6}\Big(-\frac{1}{3} + 3\Big) = \frac{\pi}{3} - \frac{4}{9}. \qquad \square$$

Problem 3. Let R be the region in the first quadrant of the plane bounded by the curves $xy - x + y = 0$ and $x - y = 1$ (see the figure below), and $f : R \to \mathbb{R}$ be defined by $f(x,y) = x^2y^2(x+y)e^{-(x-y)^2}$.



Find $\iint_R f(x,y)\, dA$ by completing the following steps.

1. (10%) Use the change of variables $xy - x + y = u$ and $x - y = v$. Find the Jacobian of x and y with respect to u and v.

2. (10%) Find the corresponding region R' of R in the uv-plane under the change of variables above.

3. (15%) Transform the double integral $\iint_R f(x,y)\, dA$ using the change of variables formula (with this particular change of variables) and compute the double integral.

Solution. 1. Since $x - y = v$, $u = xy - x + y = xy - v$; thus $xy = u + v$. Therefore,

$$(x+y)^2 = (x-y)^2 + 4xy = v^2 + 4(u+v)\,.$$

We note that $x, y \geqslant 0$ in R, $x + y = \sqrt{v^2 + 4u + 4v}$. Solving for x, y in terms of u, v, we find that

$$x = g_1(u,v) = \frac{v + \sqrt{v^2+4u+4v}}{2} \quad \text{and} \quad y = g_2(u,v) = \frac{-v + \sqrt{v^2+4u+4v}}{2}\,.$$

As a consequence,

$$x_u(u,v) = \frac{1}{\sqrt{v^2+4u+4v}} \qquad x_v(u,v) = \frac{1}{2}\Big(1 + \frac{v+2}{\sqrt{v^2+4u+4v}}\Big)\,,$$

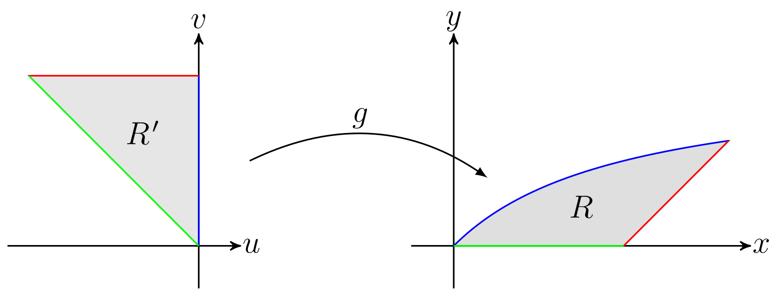
$$y_u(u,v) = \frac{1}{\sqrt{v^2+4u+4v}} \qquad y_v(u,v) = \frac{1}{2}\Big(-1 + \frac{v+2}{\sqrt{v^2+4u+4v}}\Big)\,,$$

and the above equalities imply

$$\frac{\partial(x,y)}{\partial(u,v)} = (x_u y_v - x_v y_u)(u,v) = \frac{-1}{\sqrt{v^2+4u+4v}}\,.$$

Therefore, $R = \{(u,v)\,|\,0 \leqslant v \leqslant 1, -v \leqslant u \leqslant 0\}$.

2. The curve $xy - x + y = 0$ corresponds to $u = 0$, while the lines $x - y = 1$ and $y = 0$ correspond to $v = 1$ and $u + v = 0$, respectively; thus if R' is the region enclosed by $u = 0$, $v = 1$ and $u + v = 0$, then $R = g(R')$.



3. By the change of variables formula,

$$\begin{aligned}\int_R f(x,y)\,dA &= \int_{g(R')} f(x,y)\,d(x,y) = \int_{R'} f\big(g_1(u,v), g_2(u,v)\big)\Big|\frac{\partial(x,y)}{\partial(u,v)}\Big|\,d(u,v)\\ &= \int_0^1\int_{-v}^0 (u+v)^2 e^{-v^2}\,dudv = \frac{1}{3}\int_0^1 v^3 e^{-v^2}\,dv\\ &= \frac{1}{6}\int_0^1 we^{-w}\,dw = -\frac{1}{6}(w+1)e^{-w}\Big|_{w=0}^{w=1} = -\frac{1}{6}\big(\frac{2}{e}-1\big)\,.\end{aligned}$$

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