Calculus MA1002-A Quiz 08

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Problem 1. Let $P_1 = (1, 0)$, $P_2 = (1, 1)$, $P_3 = (0, -1)$ and $P_4 = (1, -1)$ be four points on the plane. Find a straight line L so that the sum of the squared distance $S = \sum_{i=1}^{4} \text{dist}(P_i, L)^2$ is smallest, where $\text{dist}(P, L)$ denotes the distance from a point P to line L.

Solution. Assume that the line L is $\cos\theta x + \sin\theta y + k = 0$ for some θ and k. Then

$$S(\theta, k) = \sum_{i=1}^{4} (x_i \cos\theta + y_i \sin\theta + k)^2 ,$$

where $P_i = (x_i, y_i)$. We then have $\sum_{i=1}^{4} x_i^2 = 3$, $\sum_{i=1}^{4} y_i^2 = 3$, $\sum_{i=1}^{4} x_i y_i = 0$, $\sum_{i=1}^{4} x_i = 3$, $\sum_{i=1}^{4} y_i = -1$; thus

$$S(\theta, k) = 3\cos^2\theta + 3\sin^2\theta + 4k^2 + 6k\cos\theta - 2k\sin\theta = 4k^2 + 6k\cos\theta - 2k\sin\theta + 3 .$$

This implies that $S_\theta(\theta, k) = -2k\cos\theta - 6k\sin\theta$ and $S_k(\theta, k) = 6\cos\theta - 2\sin\theta + 8k$; thus the critical point of S satisfies

$$S_\theta(\theta, k) = 0 \quad \Rightarrow \quad k\cos\theta + 3k\sin\theta = 0 \quad \Rightarrow \quad k = 0 \text{ or } \cos\theta = -3\sin\theta .$$

1. If $k = 0$, then $3\cos\theta = \sin\theta$ which implies that $10\cos^2\theta = 1$. Therefore, $\cos\theta = \frac{\pm 1}{\sqrt{10}}$ and $\sin\theta = \frac{\pm 3}{\sqrt{10}}$. This gives a candidate line $x + 3y = 0$. In this case, $S(\theta, k) = 3$.
2. If $\cos\theta = -3\sin\theta$, then $\sin\theta = \frac{\pm 1}{\sqrt{10}}$ and $\cos\theta = \frac{\mp 3}{\sqrt{10}}$ which, using $S_k(\theta, k) = 0$, implies that $k = \frac{\pm 20}{8\sqrt{10}} = \frac{\pm 5}{2\sqrt{10}}$. This gives a candidate line $3x - y = \frac{5}{2}$. In this case,
$$S(\theta, k) = 4 \cdot \frac{25}{40} + 6 \cdot \frac{-15}{20} - 2 \cdot \frac{5}{20} + 3 = 0.5 .$$

Therefore, the line of interests is $3x - y = \frac{5}{2}$. □

Problem 2. (5%) Find the extreme value of $f(x, y) = xy$ subject to the constraint $x^3 - 3xy + y^3 = 1$.

Solution. Let $g(x, y) = x^3 - 3xy + y^3$. Then $(\nabla g)(x, y) = (3x^2 - 3y, -3x + 3y^2)$; thus if $(\nabla g)(x, y) = 0$, we must have

$$x^2 = y, y^2 = x \Rightarrow x^4 = x \Rightarrow x = 0 \text{ or } x = 1 \Rightarrow (x, y) = (0, 0) \text{ or } (x, y) = (1, 1) .$$

Suppose that f, subject to the constraint $g = 1$, attains its extrema at (x_0, y_0). Then by the fact that $(\nabla g)(x_0, y_0) \neq \mathbf{0}$, the Lagrange Multiplier Theorem implies that there exists $\lambda \in \mathbb{R}$ such that

$$(y_0, x_0) = (\nabla f)(x_0, y_0) = \lambda(\nabla g)(x_0, y_0) = 3\lambda(x_0^2 - y_0, -x_0 + y_0^2)$$

Therefore, $y_0 = 3\lambda(x_0^2 - y_0)$ and $x_0 = 3\lambda(-x_0 + y_0^2)$.

If $x_0 = 0$, then $(3\lambda + 1)y_0 = 0$ and $\lambda y_0^2 = 0$. Then $y_0 = 0$ which is impossible since $g(x_0, y_0) = 1$. Similarly, $\lambda \neq 0$, so $x_0, \lambda \neq 0$. Therefore,

$$y_0 \cdot 3\lambda(-x_0 + y_0^2) = 3\lambda(x_0^2 - y_0) \cdot x_0 \quad \Rightarrow \quad y_0^3 = x_0^3 \quad \Rightarrow \quad x_0 = y_0 .$$

Therefore, $g(x_0, x_0) = 2x_0^3 - x_0^2 = 1$. It is obvious that $x_0 = 1$ is a zero, and factoring shows that

$$2x_0^3 - x_0^2 - 1 = (x_0 - 1)(2x_0^2 + x_0 + 1) ;$$

hence there is no other zero. Therefore, $f(1, 1) = 1$ is an extreme value of f subject to $g = 1$. $\square$