Calculus MA1002-A Quiz 07

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Problem 1. (3%) Let R be an open region in the plane, and $f : R \to \mathbb{R}$ be a function. Write down the definition of critical points and saddle points of f.

Solution. 1. A point $(x_0, y_0) \in R$ is a critical point of f if either $f_x(x_0, y_0) = f_y(x_0, y_0) = 0$ or at least one of $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ does not exist.

2. A point $(x_0, y_0) \in R$ is a saddle point of f if (x_0, y_0) is a critical point of f but f does not attain a relative extrema at (x_0, y_0). □

Problem 2. (7%) Find the relative extrema of the function $f(x, y) = \Big(\frac{1}{2} - x^2 + y^2\Big)e^{1-x^2-y^2}$. Use the second derivative test when applicable.

Solution. We compute the first and second partial derivatives of f as follows:

$$\begin{aligned}
f_x(x, y) &= -2xe^{1-x^2-y^2} - 2x\big(\frac{1}{2} - x^2 + y^2\big)e^{1-x^2-y^2} = -2x\big(\frac{3}{2} - x^2 + y^2\big)e^{1-x^2-y^2}\,,\\
f_y(x, y) &= 2ye^{1-x^2-y^2} - 2y\big(\frac{1}{2} - x^2 + y^2\big)e^{1-x^2-y^2} = 2y\big(\frac{1}{2} + x^2 - y^2\big)e^{1-x^2-y^2}\,,\\
f_{xx}(x, y) &= \frac{\partial}{\partial x}\big(-3x + 2x^3 - 2xy^2\big)e^{1-x^2-y^2} = \big[(-3 + 6x^2 - 2y^2) - 2x(-3x + 2x^3 - 2xy^2)\big]e^{1-x^2-y^2}\\
&= \big(-3 + 12x^2 - 2y^2 - 4x^4 + 4x^2y^2\big)e^{1-x^2-y^2}\,,\\
f_{xy}(x, y) &= \frac{\partial}{\partial y}\big(-3x + 2x^3 - 2xy^2\big)e^{1-x^2-y^2} = \big[-4xy - 2y(-3x + 2x^3 - 2xy^2)\big]e^{1-x^2-y^2}\\
&= \big(2xy - 4x^3y + 4xy^3\big)e^{1-x^2-y^2}\,,\\
f_{yy}(x, y) &= \frac{\partial}{\partial y}\big(y + 2x^2y - 2y^3\big)e^{1-x^2-y^2} = \big[(1 + 2x^2 - 6y^2) - 2y(y + 2x^2y - 2y^3)\big]e^{1-x^2-y^2}\\
&= \big(1 + 2x^2 - 8y^2 - 4x^2y^2 + 4y^4\big)e^{1-x^2-y^2}\,.
\end{aligned}$$

A critical point (x_0, y_0) satisfies $x_0\big(\frac{3}{2} - x_0^2 + y_0^2\big) = y_0\big(\frac{1}{2} + x_0^2 - y_0^2\big) = 0$ which implies that

$$(x_0, y_0) = (0, 0)\,, \big(0, \pm\frac{1}{\sqrt{2}}\big)\,, \big(\pm\frac{\sqrt{3}}{\sqrt{2}}, 0\big)\,.$$

Let $D(x_0, y_0) = f_{xx}(x_0, y_0)f_{yy}(x_0, y_0) - f_{xy}(x_0, y_0)^2$.

1. If $(x_0, y_0) = (0, 0)$, then $D(x_0, y_0) < 0$ which implies that $(0, 0)$ is a saddle point of f.

2. If $(x_0, y_0) = \big(0, \pm\frac{1}{\sqrt{2}}\big)$, then $f_{xx}(x_0, y_0) = -4\sqrt{e}$, $f_{xy}(x_0, y_0) = 0$ and $f_{yy}(x_0, y_0) = -2\sqrt{e}$. Therefore, $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) < 0$; thus the second derivative test implies that f attains a relative maximum at $\big(0, \pm\frac{1}{\sqrt{2}}\big)$.

3. If $(x_0, y_0) = \left(\pm\frac{\sqrt{3}}{\sqrt{2}}, 0\right)$, then $f_{xx}(x_0, y_0) = \frac{6}{\sqrt{e}}$, $f_{xy}(x_0, y_0) = 0$ and $f_{yy}(x_0, y_0) = \frac{4}{\sqrt{e}}$. Therefore, $D(x_0, y_0) > 0$ and $f_{xx}(x_0, y_0) > 0$; thus the second derivative test implies that f attains a relative minimum at $\left(\pm\frac{\sqrt{3}}{\sqrt{2}}, 0\right)$. $\square$