Calculus MA1002-A Quiz 06

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Problem 1. (5%) Let f, g be differentiable functions of three variables. Suppose there are two differentiable functions ϕ, ψ of one variable such that $f(x,\phi(x),\psi(x)) = g(x,\phi(x),\psi(x)) = 0$. Find ϕ' and ψ' in terms of first partial derivatives of f and g if $D \equiv f_y g_z - f_z g_y \neq 0$.

Solution: By the chain rule,

$$\begin{aligned} f_x(x,\phi(x),\psi(x)) + f_y(x,\phi(x),\psi(x))\phi'(x) + f_z(x,\phi(x),\psi(x))\psi'(x) = 0\,, \\ g_x(x,\phi(x),\psi(x)) + g_y(x,\phi(x),\psi(x))\phi'(x) + g_z(x,\phi(x),\psi(x))\psi'(x) = 0\,. \end{aligned}$$

Therefore,

$$\phi'(x) = \frac{(f_z g_x - g_z f_x)(x,\phi(x),\psi(x))}{D(x,\phi(x),\psi(x))}, \quad \psi'(x) = \frac{(f_x g_y - f_y g_x)(x,\phi(x),\psi(x))}{D(x,\phi(x),\psi(x))}\,.$$

Problem 2. (5%) Let

$$f(x,y) = \begin{cases} \dfrac{x^3 y}{x^4 + y^2} & \text{if } (x,y) \neq (0,0)\,, \\ 0 & \text{if } (x,y) = (0,0)\,. \end{cases}$$

Show that the directional derivative of f at the origin exists in all directions $\boldsymbol{u} = (\cos\theta, \sin\theta)$, and

$$(D_{\boldsymbol{u}} f)(0,0) = \big(f_x(0,0), f_y(0,0)\big) \cdot (\cos\theta, \sin\theta)\,.$$

Proof. First we compute $f_x(0,0)$ and $f_y(0,0)$:

$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0) - f(0,0)}{h} = 0 \qquad \text{and} \qquad f_y(0,0) = \lim_{k\to 0} \frac{f(0,k) - f(0,0)}{k} = 0\,.$$

For $h \neq 0$,

$$\frac{f(h\cos\theta, h\sin\theta) - f(0,0)}{h} = \frac{h\cos^3\theta\sin\theta}{\sin^2\theta + h^2\cos^4\theta}\,.$$

1. If $\sin\theta = 0$, then
$$(D_{\boldsymbol{u}} f)(0,0) = \lim_{h\to 0} \frac{f(h\cos\theta, h\sin\theta) - f(0,0)}{h} = 0\,.$$
2. If $\sin\theta \neq 0$, then
$$(D_{\boldsymbol{u}} f)(0,0) = \lim_{h\to 0} \frac{f(h\cos\theta, h\sin\theta) - f(0,0)}{h} = 0\,.$$

In either cases, $(D_{\boldsymbol{u}} f)(0,0) = 0 = (f_x(0,0), f_y(0,0)) \cdot (\cos\theta, \sin\theta)$. □