Calculus MA1002-A Quiz 05

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Problem 1. (4%) Let f be defined on an open region R of the plane, and (x_0, y_0) in R. State the definition of that "f is differentiable at (x_0, y_0)".

Solution: f is differentiable at (x_0, y_0) if

$$\lim_{(x,y)\to(x_0,y_0)} \frac{\big|f(x,y) - f(x_0,y_0) - f_x(x_0,y_0)(x-x_0) - f_y(x_0,y_0)(y-y_0)\big|}{\sqrt{(x-x_0)^2+(y-y_0)^2}} = 0\,.$$

Problem 2. (6%) Let $R = \mathbb{R}^2 \backslash \big\{(x,0) \in \mathbb{R}^2 \,\big|\, x \geqslant 0\big\}$, and $f : R \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \arccos \dfrac{x}{\sqrt{x^2+y^2}} & \text{if } y > 0\,, \\ \pi & \text{if } y = 0\,, \\ 2\pi - \arccos \dfrac{x}{\sqrt{x^2+y^2}} & \text{if } y < 0\,. \end{cases}$$

Show that f is differentiable at $(a, 0)$ for all $a < 0$.

Solution: By the computation from the last quiz, we have

$$f_x(x,y) = \begin{cases} -\dfrac{y}{x^2+y^2} & \text{if } y \neq 0\,, \\ 0 & \text{if } y = 0\,, \end{cases} \qquad \text{and} \qquad f_y(x,y) = \begin{cases} \dfrac{x}{x^2+y^2} & \text{if } y \neq 0\,, \\ \dfrac{1}{x} & \text{if } y = 0\,. \end{cases}$$

Then $f_x(a,0) = 0$ and $f_y(a,0) = \dfrac{1}{a}$. Since $\big|f_x(x,y)\big| \leqslant \dfrac{|y|}{x^2+y^2}$, by the fact that $\displaystyle\lim_{(x,y)\to(a,0)} \frac{|y|}{x^2+y^2} = 0$, the Squeeze Theorem implies that

$$\lim_{(x,y)\to(a,0)} \big|f_x(x,y) - f_x(a,0)\big| = \lim_{(x,y)\to(a,0)} \big|f_x(x,y)\big| = 0\,.$$

Therefore, f_x is continuous at $(a,0)$ for all $a < 0$. On the other hand, it is clear that f_x is continuous on the upper and lower half planes; thus f_x is continuous on R.

Next, we claim that f_y is continuous on R. As the previous case, f_y is continuous on the upper and lower half planes, so we only have to focus on the continuity of f_y at $(a,0)$ for all $a < 0$. Let $a < 0$ be given. Then

$$\begin{aligned} \big|f_y(x,y) - f_y(a,0)\big| &\leqslant \Big|\frac{x}{x^2+y^2} - \frac{1}{a}\Big| + \Big|\frac{1}{x} - \frac{1}{a}\Big| = \Big|\frac{x^2+y^2-ax}{a(x^2+y^2)}\Big| + \frac{|x-a|}{|ax|} \\ &\leqslant \frac{|x-a||x| + |y|^2}{|a|(x^2+y^2)} + \frac{|x-a|}{|ax|}\,. \end{aligned}$$

Since the right-hand side function approaches 0 as (x,y) approaches $(a,0)$, the Squeeze Theorem shows that

$$\lim_{(x,y)\to(a,0)} \big|f_y(x,y) - f_y(a,0)\big| = 0\,.$$

Therefore, f_y is continuous at $(a, 0)$ for all $a < 0$; thus f_y is continuous on R.

Finally, by the continuity of f_x and f_y on R, we conclude that f is differentiable on R (so in particular, f is differentiable at $(a, 0)$ for all $a < 0$).