Calculus MA1002-A Quiz 04

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Problem 1. (2%) Let f be a function of two variables, and $(x_0, y_0) \in \mathbb{R}^2$. State the ε-δ definition of

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L\,.$$

Solution: $\forall\, \varepsilon > 0, \exists\, \delta > 0 \ni \big|f(x,y) - L\big| < \varepsilon$ whenever $0 < \sqrt{(x-x_0)^2 + (y-y_0)^2} < \delta$.

Problem 2. (8%) Let $R = \mathbb{R}^2 \backslash \big\{(x,0) \in \mathbb{R}^2 \,\big|\, x \geqslant 0\big\}$, and $f : R \to \mathbb{R}$ be given by

$$f(x,y) = \begin{cases} \arccos \dfrac{x}{\sqrt{x^2+y^2}} & \text{if } y > 0\,, \\ \pi & \text{if } y = 0\,, \\ 2\pi - \arccos \dfrac{x}{\sqrt{x^2+y^2}} & \text{if } y < 0\,. \end{cases}$$

Find the first partial derivatives of f at every point of R.

Solution: Note that $\dfrac{d}{dx}\arccos x = \dfrac{-1}{\sqrt{1-x^2}}$. On the upper half plane $y > 0$,

$$\begin{aligned} f_x(x,y) &= \frac{-1}{\sqrt{1-\frac{x^2}{x^2+y^2}}} \cdot \frac{\partial}{\partial x}\frac{x}{\sqrt{x^2+y^2}} = \frac{-1}{\sqrt{\frac{y^2}{x^2+y^2}}} \cdot \frac{\sqrt{x^2+y^2} - x\frac{\partial}{\partial x}\sqrt{x^2+y^2}}{x^2+y^2} \\ &= -\frac{\sqrt{x^2+y^2} - x\frac{x}{\sqrt{x^2+y^2}}}{|y|\sqrt{x^2+y^2}} = -\frac{y}{x^2+y^2}\,, \\ f_y(x,y) &= \frac{-1}{\sqrt{1-\frac{x^2}{x^2+y^2}}} \cdot \frac{\partial}{\partial y}\frac{x}{\sqrt{x^2+y^2}} = \frac{-1}{\sqrt{\frac{y^2}{x^2+y^2}}} \cdot \frac{-x\frac{\partial}{\partial y}\sqrt{x^2+y^2}}{x^2+y^2} \\ &= \frac{x\frac{y}{\sqrt{x^2+y^2}}}{|y|\sqrt{x^2+y^2}} = \frac{x}{x^2+y^2}\,, \end{aligned}$$

and on the lower half plane $y < 0$,

$$\begin{aligned} f_x(x,y) &= -\frac{-1}{\sqrt{1-\frac{x^2}{x^2+y^2}}} \cdot \frac{\partial}{\partial x}\frac{x}{\sqrt{x^2+y^2}} = \frac{1}{\sqrt{\frac{y^2}{x^2+y^2}}} \cdot \frac{\sqrt{x^2+y^2} - x\frac{\partial}{\partial x}\sqrt{x^2+y^2}}{x^2+y^2} \\ &= \frac{\sqrt{x^2+y^2} - x\frac{x}{\sqrt{x^2+y^2}}}{|y|\sqrt{x^2+y^2}} = -\frac{y}{x^2+y^2}\,, \\ f_y(x,y) &= -\frac{-1}{\sqrt{1-\frac{x^2}{x^2+y^2}}} \cdot \frac{\partial}{\partial y}\frac{x}{\sqrt{x^2+y^2}} = \frac{1}{\sqrt{\frac{y^2}{x^2+y^2}}} \cdot \frac{-x\frac{\partial}{\partial y}\sqrt{x^2+y^2}}{x^2+y^2} \\ &= \frac{-x\frac{y}{\sqrt{x^2+y^2}}}{|y|\sqrt{x^2+y^2}} = \frac{x}{x^2+y^2}\,. \end{aligned}$$

Now we consider the partial derivative at point $(x, 0)$ for $x < 0$. Since $f(x, 0) = \pi$ for all $x < 0$, we find that $f_x(x, 0) = 0$ for all $x < 0$. On the other hand, by the definition of partial derivatives

$$f_y(x,0) = \lim_{\Delta y \to 0} \frac{f(x,\Delta y) - f(x,0)}{\Delta y}\,.$$

We compute the limit by one-sided limit: since $\arccos(-1) = \pi$, by L'Hôspital's rule,

$$\lim_{\Delta y \to 0^+} \frac{\arccos \dfrac{x}{\sqrt{x^2+\Delta y^2}} - \pi}{\Delta y} = \lim_{\Delta y \to 0^+} \frac{\dfrac{x}{x^2+\Delta y^2}}{1} = \frac{1}{x}$$

and

$$\lim_{\Delta y \to 0^-} \frac{2\pi - \arccos \dfrac{x}{\sqrt{x^2+\Delta y^2}} - \pi}{\Delta y} = \lim_{\Delta y \to 0^-} \frac{\pi - \arccos \dfrac{x}{\sqrt{x^2+\Delta y^2}}}{\Delta y} = \lim_{\Delta y \to 0^+} \frac{\dfrac{x}{x^2+\Delta y^2}}{1} = \frac{1}{x}\,.$$

Therefore,

$$\frac{\partial f}{\partial x}(x,y) = \begin{cases} -\dfrac{y}{x^2+y^2} & \text{if } y \neq 0\,, \\ 0 & \text{if } y = 0\,, \end{cases} \qquad \text{and} \qquad \frac{\partial f}{\partial y}(x,y) = \begin{cases} \dfrac{x}{x^2+y^2} & \text{if } y \neq 0\,, \\ \dfrac{1}{x} & \text{if } y = 0\,. \end{cases}$$