Calculus MA1002-A Quiz 02

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Problem 1. (3pts) Let $a_n = \begin{cases} ne^{-n} & \text{if } n \text{ is a prime number}\,, \\ e^{-n} & \text{otherwise}\,. \end{cases}$ Does $\sum\limits_{k=1}^{\infty} a_k$ converge? Give reason for your answer.

Solution: Let $b_n = ne^{-n}$. Then $0 \leqslant a_n \leqslant b_n$ for all $n \in \mathbb{N}$. We show that $\sum\limits_{k=1}^{\infty} b_k$ converges so that by the direct comparison test, $\sum\limits_{k=1}^{\infty} a_k$ also converges.

Method 1: By the ratio test, since

$$\lim_{n\to\infty} \frac{b_{n+1}}{b_n} = \lim_{n\to\infty} \frac{(n+1)e^{-(n+1)}}{ne^{-n}} = \lim_{n\to\infty} \frac{n+1}{ne} = \frac{1}{e} < 1\,,$$

we find that $\sum\limits_{k=1}^{\infty} b_k$ converges.

Method 2: Let $f(x) = xe^{-x}$. Then

$$f'(x) = e^{-x} - xe^{-x} = (1-x)e^{-x} \leqslant 0 \qquad \forall\, x \geqslant 1\,;$$

thus f is non-negative, decreasing and continuous on $[1,\infty)$. Moreover,

$$\int_1^{\infty} xe^{-x}\,dx \leqslant \int_0^{\infty} x^{2-1}e^{-x}\,dx = \Gamma(2) < \infty\,;$$

thus $\int_1^{\infty} xe^{-x}\,dx$ converges which implies that $\sum\limits_{k=1}^{\infty} b_k$ converges. □

Problem 2. (4pts) Find all $r > 0$ such that the series $\sum\limits_{k=1}^{\infty} r^{\sqrt{k}}$ converges.

Solution: Let $a_n = r^{\sqrt{n}}$. If $r \geqslant 1$, then $\lim\limits_{n\to\infty} a_n \neq 0$; thus the n-th term test shows that $\sum\limits_{k=1}^{\infty} r^{\sqrt{k}}$ diverges if $r \geqslant 1$.

Now suppose that $0 < r < 1$. Let $f(x) = r^{\sqrt{x}}$. Then $f : [1,\infty) \to \mathbb{R}$ is non-negative, decreasing and continuous. Therefore, to see whether $\sum\limits_{k=1}^{\infty} r^{\sqrt{k}}$ converges, it suffices to look at the improper integral $\int_1^{\infty} r^{\sqrt{x}}\,dx$. Let $x = \frac{u^2}{(\ln r)^2}$. Then

$$\begin{aligned} \int_1^{\infty} r^{\sqrt{x}}\,dx &= \int_1^{\infty} e^{\sqrt{x}\ln r}\,dx = \frac{1}{(\ln r)^2}\int_{|\ln r|}^{\infty} \exp\Big(\frac{u}{|\ln r|}\ln r\Big)\cdot 2u\,du = \frac{2}{(\ln r)^2}\int_{|\ln r|}^{\infty} ue^{-u}\,du \\ &\leqslant \frac{2}{(\ln r)^2}\int_0^{\infty} ue^{-u}\,du = \frac{2}{(\ln r)^2}\Gamma(2) = \frac{2}{(\ln r)^2} < \infty\,; \end{aligned}$$

thus the improper integral $\int_1^{\infty} r^{\sqrt{x}}\,dx$ converges when $0 < r < 1$. Therefore, we conclude that $\sum\limits_{k=1}^{\infty} r^{\sqrt{k}}$ converges if and only if $0 < r < 1$. □

Problem 3. (3pts) Determine if the series $\frac{2}{5}+\frac{2\cdot 6}{5\cdot 8}+\frac{2\cdot 6\cdot 10}{5\cdot 8\cdot 11}+\frac{2\cdot 6\cdot 10\cdot 14}{5\cdot 8\cdot 11\cdot 14}+\cdots$ converges or not.

Solution: The series is $\sum_{k=1}^{\infty} a_k$, where $a_n=\frac{2\cdot 6\cdot 10\cdots(4n-2)}{5\cdot 8\cdot 11\cdots(3n+2)}$. Since

$$\lim_{n\to\infty}\frac{a_{n+1}}{a_n}=\lim_{n\to\infty}\frac{\frac{2\cdot 6\cdot 10\cdots(4n-2)(4n+2)}{5\cdot 8\cdot 11\cdots(3n+2)(3n+5)}}{\frac{2\cdot 6\cdot 10\cdots(4n-2)}{5\cdot 8\cdot 11\cdots(3n+2)}}=\lim_{n\to\infty}\frac{4n+2}{3n+5}=\frac{4}{3}>1\,,$$

the ratio test implies that $\sum_{k=1}^{\infty} a_k$ diverges. □