Calculus MA1002-A Quiz 01

National Central University, Mar. 14 2018

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Problem 1. (2pts) Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers. Write down the definition of the statement $\lim\limits_{n\to\infty} a_n = L$.

$$\lim_{n\to\infty} a_n = L \Leftrightarrow \text{for every } \varepsilon > 0 \text{ there exists } N > 0 \text{ such that } |a_n - L| < \varepsilon \text{ whenever } n \geqslant N.$$

Problem 2. Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of real numbers defined recursively by $a_{n+1} = \dfrac{1}{2 + a_n}$ with $a_1 = \dfrac{1}{2}$. Show that $\{a_n\}_{n=1}^{\infty}$ converges to $L \equiv \sqrt{2} - 1$ by completing the following:

1. (3pts) Show that $a_{n+1} - L = \dfrac{L}{2 + a_n}(L - a_n)$ for all $n \in \mathbb{N}$ and conclude that $|a_{n+1} - L| \leqslant \dfrac{L}{2}|a_n - L|$ for all $n \in \mathbb{N}$.
2. (2pts) Show that $|a_n - L| \leqslant \left(\dfrac{L}{2}\right)^{n-1}|a_1 - L|$ and conclude that $\lim\limits_{n\to\infty} a_n = L$.

Proof. First we observe $a_n \geqslant 0$ for all $n \in \mathbb{N}$. Since

$$a_{n+1} - L = \frac{1 - (\sqrt{2} - 1)(2 + a_n)}{2 + a_n} = \frac{3 - 2\sqrt{2} - (\sqrt{2} - 1)a_n}{2 + a_n} = \frac{(\sqrt{2} - 1)^2 - (\sqrt{2} - 1)a_n}{2 + a_n}$$
$$= \frac{\sqrt{2} - 1}{2 + a_n}(L - a_n)\,,$$

we find that $\big|a_{n+1} - L\big| = \dfrac{L}{2 + a_n}|a_n - L| \leqslant \dfrac{L}{2}|a_n - L|$; thus

$$\big|a_n - L\big| = \frac{L}{2}|a_{n-1} - L| \leqslant \frac{L}{2} \cdot \frac{L}{2}|a_{n-2} - L| \leqslant \cdots \leqslant \left(\frac{L}{2}\right)^{n-1}|a_1 - L|\,.$$

Since $\dfrac{L}{2} < 1$, by the squeeze theorem we find that $\lim\limits_{n\to\infty} |a_n - L| = 0$. Therefore, $\lim\limits_{n\to\infty} a_n = L$. □

Problem 3. (3pts) Determine whether the series $\sum\limits_{k=1}^{\infty} \ln \dfrac{k}{2k+1}$ converges or not.

Proof. Since $\lim\limits_{n\to\infty} \ln \dfrac{n}{2n+1} = \ln \dfrac{1}{2} = -\ln 2 \neq 0$, we find that the series $\sum\limits_{k=1}^{\infty} \ln \dfrac{k}{2k+1}$ must diverge. □