Extra Exercise Problem Set 8

Mar. 14 2019

Problem 1. Let $\{a_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of real numbers such that $a_n, b_n > 0$ for all $n \geqslant N$. Define

$$c_n = b_n - b_{n+1}\frac{a_{n+1}}{a_n} \qquad \forall\, n \in \mathbb{N}\,. \tag{$\star$}$$

1. Show that if there exists a constant $r > 0$ such that $r < c_n$ for all $n \geqslant N$, then $\sum\limits_{k=1}^{\infty} a_k$ converges.

   Hint: Rewrite ($\star$) as $b_n = c_n + \frac{a_{n+1}}{a_n}b_{n+1}$ and then obtain

$$\begin{aligned} b_N &= c_N + \frac{a_{N+1}}{a_N}b_{N+1} = c_N + \frac{a_{N+1}}{a_N}\Big(c_{N+1} + \frac{a_{N+2}}{a_{N+1}}b_{N+2}\Big) = c_N + \frac{a_{N+1}}{a_N}c_{N+1} + \frac{a_{N+2}}{a_N}b_{N+2} \\ &= c_N + \frac{a_{N+1}}{a_N}c_{N+1} + \frac{a_{N+2}}{a_N}\Big(c_{N+2} + \frac{a_{N+3}}{a_{N+2}}b_{N+3}\Big) = \cdots \\ &= c_N + \frac{a_{N+1}}{a_N}c_{N+1} + \frac{a_{N+2}}{a_N}c_{N+2} + \cdots + \frac{a_{N+n}}{a_N}c_{N+n} + \frac{a_{N+n+1}}{a_N}b_{N+n+1}\,. \end{aligned}$$

   Use the fact that $0 < r < c_n$ for all $n \geqslant N$ to conclude that

$$\sum_{k=N}^{N+n} a_k \leqslant \frac{a_N b_N}{r} \qquad \forall\, n \in \mathbb{N}\,.$$

   Note that then the sequence of partial sum of $\sum\limits_{k=1}^{\infty} a_k$ then is bounded from above $\Big($by $\sum\limits_{k=1}^{N-1} a_k + \frac{a_N b_N}{r}\Big)$.

2. Show that if $\sum\limits_{k=1}^{\infty} \frac{1}{b_k}$ diverges and $c_n \leqslant 0$ for all $n \geqslant N$, then $\sum\limits_{k=1}^{\infty} a_k$ diverges.

   Hint: The fact that $c_n \leqslant 0$ for all $n \geqslant N$ implies that $b_n a_n \leqslant b_{n+1}a_{n+1}$ for all $n \geqslant N$. Use this fact to conclude that

$$\frac{a_N b_N}{b_n} \leqslant a_n \qquad \forall\, n \geqslant N$$

   and then apply the direct comparison test to conclude that $\sum\limits_{k=1}^{\infty} a_k$ diverges.

There are other tests of convergence of infinite series that we will not mention in class. In this exercise problem set, we list some of them and you should try to understand how they are proved and how they can be applied.

Problem 2. Let $\sum\limits_{k=1}^{\infty} a_k$ be a series with positive terms, and $\lim\limits_{n\to\infty} \frac{a_{n+1}}{a_n} = 1$. We know from class that the ratio test fails when this happens, but there are some refined results concerning this particular case.

1. **(Raabe's test)**:

   (a) If there exists a constant $\mu > 1$ such that $\frac{a_{n+1}}{a_n} < 1 - \frac{\mu}{n}$ for all $n \geqslant N$, then $\sum\limits_{k=1}^{\infty} a_k$ converges.

(b) If there exists a constant $0 < \mu < 1$ such that $\dfrac{a_{n+1}}{a_n} > 1 - \dfrac{\mu}{n}$ for all $n \geqslant N$, then $\sum\limits_{k=1}^{\infty} a_k$ diverges.

Hint: Consider the sequence $\{b_n\}_{n=1}^{\infty}$ defined by $b_n = (n-1)a_n - na_{n+1}$. Then $\sum\limits_{k=1}^{\infty} b_k$ is a telescoping series. For case (a), show that $\big\{na_{n+1}\big\}_{n=N}^{\infty}$ is a positive decreasing sequence and then conclude that $\sum\limits_{k=1}^{\infty} b_k$ converges. Note that $b_n \geqslant (\mu - 1)a_n$ for all $n \geqslant N$. For case (b), show that $\big\{na_{n+1}\big\}_{n=N}^{\infty}$ is a positive increasing sequence; thus $a_n \geqslant \dfrac{Na_{N+1}}{n-1}$ for all $n \geqslant N+1$ which implies that $\sum\limits_{k=1}^{\infty} a_k$ diverges.

Remark: 注意到 (a) 說的是如果 $\{a_n\}_{n=1}^{\infty}$ 在某項之後「遞減得夠快」，那麼 $\sum\limits_{k=1}^{\infty} a_k$ 收斂。反之，如果 $\{a_n\}_{n=1}^{\infty}$「並非遞減得那麼快」，那麼 $\sum\limits_{k=1}^{\infty} a_k$ 發散。

2. **(Gauss's test)** - left to next week (Mar. 26): Suppose that there exist a positive constant $\epsilon > 0$, a constant μ, and a **bounded** sequence $\{R_n\}_{n=1}^{\infty}$ such that
$$\frac{a_{n+1}}{a_n} = 1 - \frac{\mu}{n} + \frac{R_n}{n^{1+\epsilon}} \qquad \text{for all } n \geqslant N\,.$$
(a) If $\mu > 1$, then $\sum\limits_{k=1}^{\infty} a_k$ converges. (b) If $\mu \leqslant 1$, then $\sum\limits_{k=1}^{\infty} a_k$ diverges.

Hint: Show that if $\mu > 1$ or $\mu < 1$, one can apply Raabe's test to conclude Gauss's test. For the case $\mu = 1$, let $b_n = (n-1)\log(n-1)$ for $n \geqslant 2$. Using the second result of Problem 1 to show the divergence of $\sum\limits_{k=1}^{\infty} a_k$ (by showing that c_n defined by ($\star$) is non-positive for all large enough n).

Problem 3. Complete the following.

1. Show that $\sum\limits_{k=1}^{\infty} \Big(1 - \dfrac{1}{\sqrt{k}}\Big)^k$ converges.
2. Show that $\sum\limits_{k=2}^{\infty} \dfrac{\log(k+1) - \log k}{(\log k)^2}$ converges.
3. - left to next week (Mar. 26) Use Gauss's test to show that both the general harmonic series $\sum\limits_{k=1}^{\infty} \dfrac{1}{ak+b}$, where $a \neq 0$, and the series $\sum\limits_{k=1}^{\infty} \dfrac{1}{\sqrt{k}}$ diverge.
4. - left to next week (Mar. 26) Show that $\sum\limits_{k=1}^{\infty} \dfrac{k!}{(\alpha+1)(\alpha+2)\cdots(\alpha+k)}$ converges if $\alpha > 1$ and diverges if $\alpha \leqslant 1$.
5. - left to next week (Mar. 26) Test the following "hypergeometric" series for convergence or divergence:

(a) $\displaystyle\sum_{k=1}^{\infty} \frac{\alpha(\alpha+1)(\alpha+2)\cdots(\alpha+k-1)}{\beta(\beta+1)(\beta+2)\cdots(\beta+k-1)} = \frac{\alpha}{\beta} + \frac{\alpha(\alpha+1)}{\beta(\beta+1)} + \frac{\alpha(\alpha+1)(\alpha+2)}{\beta(\beta+1)(\beta+2)} + \cdots.$

(b) $\displaystyle 1 + \frac{\alpha\cdot\beta}{1\cdot\gamma} + \frac{\alpha(\alpha+1)\cdot\beta(\beta+1)}{1\cdot 2\gamma\cdot(\gamma+1)} + \frac{\alpha(\alpha+1)(\alpha+2)\cdot\beta(\beta+1)(\beta+2)}{1\cdot 2\cdot 3\cdot\gamma(\gamma+1)(\gamma+2)} + \cdots.$