Extra Exercise Problem Set 15

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Problem 1. In this problem we try to prove the Lagrange Multiplier Theorem when there are two constraints. Let f, g, h be continuously differentiable functions of three variables. Suppose that subject to the constraints $g(x, y, z) = h(x, y, z) = c$, the function f attains its extrema at (x_0, y_0, z_0), and $(\nabla g)(x_0, y_0, z_0) \times (\nabla h)(x_0, y_0, z_0) \neq \mathbf{0}$. Complete the following.

1. Assume that the first component of $(\nabla g)(x_0, y_0, z_0) \times (\nabla h)(x_0, y_0, z_0)$ is non-zero (then in a neighborhood of (x_0, y_0, z_0) this cross product is also non-zero). By a more general implicit function theorem, there exist $\delta > 0$ and two continuously differentiable functions $\phi, \psi : (x_0 - \delta, x_0 + \delta) \to \mathbb{R}$ such that

$$g\big(x, \phi(x), \psi(x)\big) = h\big(x, \phi(x), \psi(x)\big) = c\,.$$

   Find ϕ' and ψ' in terms of partial derivatives of g and h.

2. Let $G(x) = f\big(x, \phi(x), \psi(x)\big)$. Then $G : (x_0 - \delta, x_0 + \delta)$ attains its extrema at x_0. Deduce that

$$(\nabla f)(x_0, y_0, z_0) \cdot \big[(\nabla g)(x_0, y_0, z_0) \times (\nabla h)(x_0, y_0, z_0)\big] = 0\,. \qquad (\star)$$

3. Use $(\star)$ to conclude that there exists $\lambda, \mu \in \mathbb{R}$ such that

$$(\nabla f)(x_0, y_0, z_0) = \lambda(\nabla g)(x_0, y_0, z_0) + \mu(\nabla h)(x_0, y_0, z_0)\,.$$

Problem 2. Let $f(x, y) = x^2 + 6(y^2 + y + 1)^2$ and $g(x, y) = x^2 + (y^3 - 1)^2$. Use the method of Lagrange Multipliers to find the extreme value of f subject to the constraint $g = 1$.