Extra Exercise Problem Set 14

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Problem 1. In this example we prove the second derivatives test for functions of two variables. Let $R \subseteq \mathbb{R}^2$ be an open region, and $f : R \to \mathbb{R}$ be a function of two variables. Suppose that f has continuous second partial derivatives. For $(x,y),(a,b) \in R$ such that the line segment connecting (x,y) and (a,b) lies in R (that is, $\gamma(t) \equiv \big(a+t(x-a), b+t(y-b)\big) \subseteq R$ for all $t \in [0,1]$), in previous exercise we have show that there exists $(\xi_1,\xi_2) = \big(a+t_0(x-a), b+t_0(y-b)\big)$, where $0 < t_0 < 1$, such that

$$\begin{aligned} f(x,y) &= f(a,b) + f_x(a,b)(x-a) + f_y(a,b)(y-b) \\ &\quad + \frac{1}{2}\big[f_{xx}(\xi_1,\xi_2)(x-a)^2 + 2f_{xy}(\xi_1,\xi_2)(x-a)(y-b) + f_{yy}(\xi_1,\xi_2)(y-b)^2\big] \end{aligned} \tag{$\star$}$$

Let $d(x,y) = f_{xx}(x,y)f_{yy}(x,y) - f_{xy}(x,y)^2$.

1. Show that if $d(a,b) > 0$ and $f_{xx}(a,b) > 0$, then there exists $\delta > 0$ such that

   $$d(x,y) > 0 \;\text{ and }\; f_{xx}(x,y) > 0 \quad \text{for all } (x,y) \in D\big((a,b),\delta\big).$$

   Therefore, if $(x,y) \in D\big((a,b),\delta\big)$, the point (ξ_1,ξ_2) validating $(\star)$ satisfies $d(\xi_1,\xi_2) > 0$ and $f_{xx}(\xi_1,\xi_2) > 0$. Similarly, show that if $d(\xi_1,\xi_2) > 0$ and $f_{xx}(\xi_1,\xi_2) < 0$, then there exists $\delta > 0$ such that

   $$d(x,y) > 0 \;\text{ and }\; f_{xx}(x,y) < 0 \quad \text{for all } (x,y) \in D\big((a,b),\delta\big).$$

   Therefore, if $(x,y) \in D\big((a,b),\delta\big)$, the point (ξ_1,ξ_2) validating $(\star)$ satisfies $d(\xi_1,\xi_2) > 0$ and $f_{xx}(\xi_1,\xi_2) < 0$.

2. Show that if $d(\xi_1,\xi_2) > 0$ and $f_{xx}(\xi_1,\xi_2) < 0$, then

   $$f_{xx}(\xi_1,\xi_2)(x-a)^2 + 2f_{xy}(\xi_1,\xi_2)(x-a)(y-b) + f_{yy}(\xi_1,\xi_2)(y-b)^2 \leqslant 0 \quad \forall\,(x,y) \in D\big((a,b),\delta\big).$$

   Similarly, show that if $d(\xi_1,\xi_2) > 0$ and $f_{xx}(\xi_1,\xi_2) > 0$, then

   $$f_{xx}(\xi_1,\xi_2)(x-a)^2 + 2f_{xy}(\xi_1,\xi_2)(x-a)(y-b) + f_{yy}(\xi_1,\xi_2)(y-b)^2 \geqslant 0 \quad \forall\,(x,y) \in D\big((a,b),\delta\big).$$

3. Use 2 to conclude the second derivatives test:

   (a) If (a,b) is a critical point of f and $d(a,b) > 0$, $f_{xx}(a,b) > 0$, then f attains a relative minimum at (a,b).

   (b) If (a,b) is a critical point of f and $d(a,b) > 0$, $f_{xx}(a,b) < 0$, then f attains a relative maximum at (a,b).