Extra Exercise Problem Set 13

May. 3 2019

Problem 1. Let $R \subseteq \mathbb{R}^2$ be an open region, and $f : R \to \mathbb{R}$ be a function of two variables. Suppose that all the k-th partial derivatives of f are continuous for $0 \leqslant k \leqslant n+1$. For $(x, y), (a, b) \in R$, let $\gamma(t) = t(x, y) + (1-t)(a, b)$ and define $g(t) = (f \circ \gamma)(t) = f\big(a + t(x-a), b + t(y-b)\big)$. Assume that $\gamma(t) \in R$ for all $t \in [0, 1]$.

1. Show (by induction) that for $1 \leqslant k \leqslant n+1$,

$$g^{(k)}(t) = \sum_{j=0}^{k} C_j^k \frac{\partial^k f}{\partial x^{k-j} \partial y^j}\big(a + t(x-a), b + t(y-b)\big)(x-a)^{k-j}(y-b)^j \,. \qquad (\star)$$

   You may need Pascal's theorem $C_j^\ell + C_{j-1}^\ell = C_j^{\ell+1}$ for all $1 \leqslant j \leqslant \ell$.

2. Show (by Taylor's theorem) that

$$f(x, y) = \sum_{k=0}^{n} \frac{1}{k!} \sum_{j=0}^{k} C_j^k \frac{\partial^k f}{\partial x^{k-j} \partial y^j}(a, b)(x-a)^{k-j}(y-b)^j + R_n(x, y)\,, \qquad (\star\star)$$

   Note that $g(1) = f(x, y)$ and $g(0) = f(a, b)$.

3. The function $\displaystyle\sum_{k=0}^{n} \frac{1}{k!} \sum_{j=0}^{k} C_j^k \frac{\partial^k f}{\partial x^{k-j} \partial y^j}(a, b)(x-a)^{k-j}(y-b)^j$ is called the n-th Taylor polynomial for f at (a, b). Write down the second and third Taylor polynomial without using the Σ notation.

4. Find the third Taylor polynomial for $f(x, y) = \exp(x^2 + 2y)$ and $g(x, y) = \sin(xy)$ at $(0, 0)$.