微積分 MA1001-A 上課筆記（精簡版）

2018.12.27.

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Theorem 8.5: Integration by Parts

If u and v are functions of x and have continuous derivatives, then

$$\int u\,dv = uv - \int v\,du\,.$$

Using integration by parts, we have shown the following formulas:

$$\int x^r \ln x\,dx = \begin{cases} \dfrac{1}{r+1}x^{r+1}\ln x - \dfrac{1}{(r+1)^2}x^{r+1} + C & \text{if } r \neq -1\,, \\ \dfrac{1}{2}(\ln x)^2 + C & \text{if } r = -1\,. \end{cases}$$

as well as

$$\begin{aligned}
\int e^{ax}\sin(bx)\,dx &= \frac{1}{a^2+b^2}\big[ae^{ax}\sin(bx) - be^{ax}\cos(bx)\big] + C\,, \\
\int e^{ax}\cos(bx)\,dx &= \frac{1}{a^2+b^2}\big[ae^{ax}\cos(bx) + be^{ax}\sin(bx)\big] + C\,, \\
\int x^n e^{ax}\,dx &= \frac{1}{a}x^n e^{ax} - \int \frac{1}{a}e^{ax}\cdot nx^{n-1}\,dx = \frac{1}{a}x^n e^{ax} - \frac{n}{a}\int x^{n-1}e^{ax}\,dx\,, \\
\int x^n \sin(ax)\,dx &= -\frac{1}{a}x^n\cos(ax) + \frac{n}{a^2}x^{n-1}\sin(ax) - \frac{n(n-1)}{a^2}\int x^{n-2}\cos(ax)\,dx\,, \\
\int x^n \cos(ax)\,dx &= \frac{1}{a}x^n\sin(ax) + \frac{n}{a^2}x^{n-1}\cos(ax) - \frac{n(n-1)}{a^2}\int x^{n-2}\sin(ax)\,dx\,, \\
\int \cos^n x\,dx &= \frac{\sin x\cos^{n-1}x}{n} + \frac{n-1}{n}\int \cos^{n-2}x\,dx\,, \\
\int \sin^n x\,dx &= -\frac{\cos x\sin^{n-1}x}{n} + \frac{n-1}{n}\int \sin^{n-2}x\,dx\,.
\end{aligned}$$

Theorem 8.15: Wallis's Formulas

If n is a non-negative integer, then

$$\int_0^{\frac{\pi}{2}} \sin^{2n+1}x\,dx = \int_0^{\frac{\pi}{2}} \cos^{2n+1}x\,dx = \frac{(2^n n!)^2}{(2n+1)!}$$

and

$$\int_0^{\frac{\pi}{2}} \sin^{2n}x\,dx = \int_0^{\frac{\pi}{2}} \cos^{2n}x\,dx = \frac{(2n)!}{(2^n n!)^2}\cdot\frac{\pi}{2}\,.$$

8.3.1 The integral of $\sin^m x \cos^n x$

- **The case when one of m and n is odd**

$$\begin{aligned}\int \sin^{2k+1} x \cos^n x \, dx &= -\int \cos^n x (1 - \cos^x x)^k \, d(\cos x) \,, \\ \int \sin^m x \cos^{2\ell+1} x \, dx &= \int \sin^m x (1 - \sin^2 x)^\ell \, d(\sin x) \,.\end{aligned}$$

- **The case when m and n are both even**

$$\begin{aligned}\int \cos^{2\ell+1} x \, dx &= \int (1 - \sin^2 x)^\ell d(\sin x) \,, \\ \int \cos^{2\ell} x \, dx &= \int \Big(\frac{1 + \cos(2x)}{2}\Big)^\ell \, dx \,, \\ \int \sin^{2k} x \cos^{2\ell} x \, dx &= \frac{1}{2^{k+\ell}} \int \big(1 - \cos(2x)\big)^k \big(1 + \cos(2x)\big)^\ell \, dx \,.\end{aligned}$$

8.3.2 The integral of $\sec^m x \tan^n x$

- **The case when m is even**

$$\begin{aligned}\int \tan^n x \, dx &= \frac{1}{n-1} \tan^{n-1} x - \int \tan^{n-2} x \, dx \,, \\ \int \sec^{2k} x \tan^n x \, dx &= \int (1 + \tan^2 x)^{k-1} \tan^n x \, d(\tan x) \,.\end{aligned}$$

- **The case when n is odd**

Suppose that $n = 2\ell + 1$ is odd and $m \geqslant 1$. Then

$$\int \sec^m x \tan^{2\ell+1} x \, dx = \int \sec^{m-1} x \tan^{2\ell} \sec x \tan x \, dx = \int \sec^{m-1} x (\sec^2 x - 1)^\ell \, d(\sec x)$$

which can be obtained by integrating polynomials.

- **The case when m is odd and n is even**

Suppose that $m = 2k + 1$ and $n = 2\ell$. Then

$$\int \sec^{2k+1} x \tan^{2\ell} x \, dx = \int \sec^{2k+1} x (\sec^2 x + 1)^\ell \, dx \,;$$

thus it suffices to know how to compute $\displaystyle\int \sec^m x\,dx$.

Using integration by parts,

$$\begin{aligned}\int \sec^m x\,dx = \int \sec^{m-2} x\,d(\tan x) &= \tan x \sec^{m-2} x - \int \tan x\,d(\sec^{m-2} x)\\ &= \tan x \sec^{m-2} x - (m-2)\int \tan^2 x \sec^{m-2} x\,dx\\ &= \tan x \sec^{m-2} x - (m-2)\int (\sec^2 x - 1)\sec^{m-2} x\,dx\end{aligned}$$

thus rearranging terms we obtain the recurrence relation

$$\int \sec^m x\,dx = \frac{m-2}{m-1}\tan x \sec^{m-2} x + \frac{m-2}{m-1}\int \sec^{m-2} x\,dx\,.$$

Example 8.19. Find the indefinite integral $\displaystyle\int \sec^4(3x)\tan^3(3x)\,dx$.

By the discussion above,

$$\begin{aligned}\int \sec^4(3x)\tan^3(3x)\,dx &= \frac{1}{3}\int \sec^2(3x)\tan^3(3x)d(\tan(3x))\\ &= \frac{1}{3}\int \big[\tan^2(3x)+1\big]\tan^3(3x)d(\tan(3x))\\ &= \frac{1}{3}\Big[\frac{1}{6}\tan^6(3x) + \frac{1}{4}\tan^4(3x)\Big] + C\,.\end{aligned}$$

8.3.3 Other techniques of integration involving trigonometric functions

- Integration by substitution (for integrand with special structures):

Example 8.20. Find the indefinite integral $\displaystyle\int \frac{\cos^3 x}{\sqrt{\sin x}}\,dx$.

Let $u = \sin x$. Then $du = \cos x\,dx$; thus

$$\begin{aligned}\int \frac{\cos^3 x}{\sqrt{\sin x}}\,dx &= \int \frac{(1-u^2)}{\sqrt{u}}\,du = \int \big(u^{-\frac{1}{2}} - u^{\frac{3}{2}}\big)\,du\\ &= \frac{1}{1-\frac{1}{2}}u^{\frac{1}{2}} - \frac{1}{1+\frac{3}{2}}u^{\frac{5}{2}} + C = 2\sqrt{\sin x} - \frac{5}{2}\sin^{\frac{5}{2}} x + C\,.\end{aligned}$$

Example 8.21. Find the indefinite integral $\displaystyle\int \frac{\sec x}{\tan^2 x}\,dx$.

Rewrite the integrand into a fraction of sine and cosine, we find that

$$\int \frac{\sec x}{\tan^2 x}\,dx = \int \frac{\cos x}{\sin^2 x}\,dx = \int \frac{1}{\sin^2 x}\,d(\sin x) = -\sin^{-1} x + C = -\csc x + C\,.$$

Example 8.22. Find the indefinite integral $\int \frac{\tan^3 x}{\sqrt{\sec x}}\,dx$.

Let $u = \sec x$. Then $du = \sec x \tan x\,dx$; thus

$$\begin{aligned}\int \frac{\tan^3 x}{\sqrt{\sec x}}\,dx &= \int \frac{(\sec^2 x - 1)\sec x \tan x}{\sec^{\frac{3}{2}} x}\,dx = \int \frac{u^2-1}{u^{\frac{3}{2}}}\,du = \int \big(u^{\frac{1}{2}} - u^{-\frac{3}{2}}\big)\,du \\ &= \frac{2}{3}u^{\frac{3}{2}} + 2u^{-\frac{1}{2}} + C = \frac{2}{3}\sec^{\frac{3}{2}} x + 2\cos^{\frac{1}{2}} x + C\,.\end{aligned}$$

• When the angular variable are different, making use of the sum and difference formula:

Example 8.23. Find the indefinite integral $\int \sin^3(5x)\cos(4x)\,dx$.

Using the sum and difference formula

$$\sin\theta\cos\phi = \frac{1}{2}\big[\sin(\theta+\phi) + \sin(\theta-\phi)\big]\,, \quad \sin\theta\sin\phi = \frac{1}{2}\big[\cos(\theta-\phi) - \sin(\theta+\phi)\big]\,,$$

we find that

$$\begin{aligned}\int \sin^3(5x)\cos(4x)\,dx &= \frac{1}{2}\int \sin^2(5x)\big[\sin(9x) + \sin x\big]\,dx \\ &= \frac{1}{4}\int \sin(5x)\big[\cos(4x) - \cos(14x) + \cos(4x) - \cos(6x)\big]\,dx \\ &= \frac{1}{8}\int \big[2\sin(9x) + 2\sin x - \sin(19x) + \sin(9x) - \sin(11x) + \sin x\big]\,dx \\ &= \frac{1}{8}\Big[-\frac{1}{3}\cos(9x) - 3\cos x + \frac{1}{19}\cos(19x) + \frac{1}{11}\cos(11x)\Big] + C\,.\end{aligned}$$

8.4 Trigonometric Substitution

We have talked about the trigonometric substitution such as

1. letting $u = a\sin x$ when seeing $\sqrt{a^2 - x^2}$;
2. letting $u = a\sec x$ when seeing $\sqrt{x^2 - a^2}$;
3. letting $u = a\tan x$ when seeing $a^2 + x^2$ in the denominator,

so we only focus on the integral $\int \frac{dx}{(a^2+x^2)^n}$, where n is a positive integer.

Let $x = a\tan u$. Then $dx = a\sec^2 u\,du$; thus

$$\int \frac{dx}{(a^2+x^2)^n} = \int \frac{a\sec^2 u}{\big[a^2(\tan^2 u + 1)\big]^n}\,du = a^{1-2n}\int \cos^{2n-2} u\,du$$

which we know how to compute.

8.5 Partial Fractions - 部份分式

In this section, we are concerned with the integrals $\int \frac{N(x)}{D(x)}\,dx$, where N, D are polynomial functions.

Write $N(x) = D(x)Q(x) + R(x)$, where Q, R are polynomials such that the degree of R is less than the degree of D (such an R is called a remainder). Then $\frac{N(x)}{D(x)} = R(x) + \frac{R(x)}{D(x)}$. Since it is easy to find the indefinite integral of R, it suffices to consider the case when the degree of the numerator is less than the degree of the denominator.

W.L.O.G., we assume that N and D no common factor, $\deg(N) < \deg(D)$, and the leading coefficient of D is 1. Since D is a polynomial with real coefficients,

$$D(x) = \Big(\prod_{j=1}^{m}(x+q_j)^{r_j}\Big)\Big(\prod_{j=1}^{n}(x^2+b_jx+c_j)^{d_j}\Big),$$

where $r_j, d_j \in \mathbb{N}$, $q_j \neq q_k$ for all $j \neq k$, $b_j \neq b_k$ or $c_j \neq c_k$ for all $j \neq k$, and $b_j^2 - 4c_j < 0$ for all $1 \leqslant j \leqslant m$. In other words, $-q_j$ are zeros of D with multiplicity r_j, and $\frac{-b_j \pm i\sqrt{4c_j - b_j^2}}{2}$ are zeros of D with multiplicity d_j, here $i = \sqrt{-1}$.