微積分 MA1001-A 上課筆記（精簡版）

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Definition 5.8

The function $\ln : (0, \infty) \to \mathbb{R}$ is defined by

$$\ln x = \int_1^x \frac{1}{t}\, dt \qquad \forall\, x > 0\,.$$

- $\ln : (0, \infty) \to \mathbb{R}$. is one-to-one and onto.

Definition 5.25

The natural exponential function $\exp : \mathbb{R} \to (0, \infty)$ is a function defined by

$$\exp(x) = y \qquad \text{if and only if} \qquad x = \ln y\,.$$

Definition 5.26

Let $a > 0$ be a real number. For each $x \in \mathbb{R}$, the exponential function to the base a, denote by $y = a^x$, is defined by $a^x \equiv \exp(x \ln a)$. In other words,

$$a^x = \exp(x \ln a) \qquad \forall\, x \in \mathbb{R}\,.$$

- **The range and the strict monotonicity of the exponential functions**

The exponential function to the base a is a strictly decreasing function if $a > 1$, while the exponential function to the base a is a strictly decreasing function if $0 < a < 1$. Moreover, for $0 < a \neq 1$, the exponential function $a^{\cdot} : \mathbb{R} \to (0, \infty)$ is one-to-one and onto.

Corollary 5.33

For $a > 0$, $\dfrac{d}{dx} a^x = a^x \ln a$ for all $x \in \mathbb{R}$ $\Big($so $\displaystyle\int a^x\, dx = \frac{a^x}{\ln a} + C\Big)$.

Definition 5.38

Let $0 < a \neq 1$ be a real number. The logarithmic function to the base a, denoted by $\log_a$, is the inverse function of the exponential function to the base a. In other words,

$$y = \log_a x \qquad \text{if and only if} \qquad a^y = x\,.$$

Theorem 5.39

Let $0 < a \neq 1$. Then $\log_a x = \dfrac{\ln x}{\ln a}$ for all $x > 0$.

By Theorem 5.39, we find that $\dfrac{d}{dx}\log_a x = \dfrac{1}{x\ln a} \qquad \forall\, x > 0\,.$

5.6 Indeterminate Forms and L'Hôspital's Rule

Theorem 5.41: Cauchy Mean Value Theorem

Let $f, g : [a, b] \to \mathbb{R}$ be continuous on $[a, b]$ and differentiable on (a, b). If $g'(x) \neq 0$ for all $x \in (a, b)$, then there exists $c \in (a, b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}\,.$$

Proof. Let $h : [a, b] \to \mathbb{R}$ be defined by

$$h(x) = \big(f(x) - f(a)\big)\big(g(b) - g(a)\big) - \big(f(b) - f(a)\big)\big(g(x) - g(a)\big)\,.$$

Then $h(a) = h(b) = 0$, and h is differentiable on (a, b). Then Rolle's Theorem implies that there exists $c \in (a, b)$ such that $h'(c) = 0$; thus for some $c \in (a, b)$,

$$f'(c)\big(g(b) - g(a)\big) - \big(f(b) - f(a)\big)g'(c) = 0\,.$$

Since $g'(x) \neq 0$ for all $x \in (a, b)$, the Mean Value Theorem implies that $g(b) \neq g(a)$. Therefore, the equality above implies that

$$\frac{f'(c)}{g'(c)} = \frac{f(b) - f(a)}{g(b) - g(a)}$$

for some $c \in (a, b)$. □

Theorem 5.42: L'Hôspital's Rule

Let f, g be differentiable on (a, b), and $\dfrac{f(x)}{g(x)}$ and $\dfrac{f'(x)}{g'(x)}$ be defined on (a, b). If $\lim\limits_{x\to a^+} \dfrac{f'(x)}{g'(x)}$ exists, and one of the following conditions holds:

1. $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^+} g(x) = 0;$
2. $\lim\limits_{x\to a^+} f(x) = \lim\limits_{x\to a^+} g(x) = \infty,$

then $\lim\limits_{x\to a^+} \dfrac{f(x)}{g(x)}$ exists, and

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}\,.$$

Proof. We first prove L'Hôspital's rule for the case that $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = 0$. Define $F, G : (a, b) \to \mathbb{R}$ by

$$F(x) = \begin{cases} f(x) & \text{if } x \in (a,b)\,, \\ 0 & \text{if } x = a\,, \end{cases} \qquad \text{and} \qquad G(x) = \begin{cases} g(x) & \text{if } x \in (a,b)\,, \\ 0 & \text{if } x = a\,. \end{cases}$$

Then for all $x \in (a, b)$, F, G are continuous on the closed $[a, x]$, and differentiable on the open interval with end-points (a, x). Therefore, the Cauchy Mean Value Theorem implies that there exists a point c between a and x such that

$$\frac{f'(c)}{g'(c)} = \frac{F'(c)}{G'(c)} = \frac{F(x) - F(a)}{G(x) - G(a)} = \frac{F(x)}{G(x)} = \frac{f(x)}{g(x)}\,.$$

Since c approaches a as x approaches a, we have

$$\lim_{x\to a^+} \frac{f'(c)}{g'(c)} = \lim_{c\to a^+} \frac{f'(c)}{g'(c)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}\,;$$

thus

$$\lim_{x\to a^+} \frac{f(x)}{g(x)} = \lim_{x\to a^+} \frac{f'(c)}{g'(c)} = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}\,.$$

Next we prove L'Hôspital's rule for the case that $\lim_{x\to a^+} f(x) = \lim_{x\to a^+} g(x) = \infty$. Let $L = \lim_{x\to a^+} \frac{f'(x)}{g'(x)}$ and $\varepsilon > 0$ be given. Then there exists $\delta_1 > 0$ such that

$$\Big|\frac{f'(x)}{g'(x)} - L\Big| < \frac{\varepsilon}{2} \qquad \text{whenever} \quad a < x < a + \delta_1\,(< b)\,.$$

Let $d = a + \delta_1$. For $a < x < d$, the Cauchy mean value theorem implies that for some c in (x, d) such that

$$\frac{f(x) - f(d)}{g(x) - g(d)} = \frac{f'(c)}{g'(c)}\,.$$

Note that the quotient above belongs to $\big(L - \frac{\varepsilon}{2}, L + \frac{\varepsilon}{2}\big)$ (if $a < x < d$). Moreover,

$$\frac{f(x) - f(d)}{g(x) - g(d)} - \frac{f(x)}{g(x)} = \frac{\big(f(x) - f(d)\big)g(d) - \big(g(x) - g(d)\big)f(d)}{\big(g(x) - g(d)\big)g(x)} = \frac{f'(c)}{g'(c)}\frac{g(d)}{g(x)} - \frac{f(d)}{g(x)}\,;$$

thus

$$\Big|\frac{f(x) - f(d)}{g(x) - g(d)} - \frac{f(x)}{g(x)}\Big| \leqslant \Big(|L| + \frac{\varepsilon}{2}\Big)\Big|\frac{g(d)}{g(x)}\Big| + \Big|\frac{f(d)}{g(x)}\Big| \qquad \text{whenever} \quad a < x < d\,.$$

Since $\lim_{x\to a^+} g(x) = \infty$, the right-hand side of the inequality above approaches zero as x approaches a from the right. Therefore, there exists $0 < \delta < \delta_1$, such that

$$\Big|\frac{f(x)-f(d)}{g(x)-g(d)} - \frac{f(x)}{g(x)}\Big| < \frac{\varepsilon}{2} \qquad \text{whenever} \quad a < x < a+\delta\,(< d < b)\,.$$

Therefore, if $a < x < a+\delta$,

$$\Big|\frac{f(x)}{g(x)} - L\Big| \leqslant \Big|\frac{f(x)-f(d)}{g(x)-g(d)} - \frac{f(x)}{g(x)}\Big| + \Big|\frac{f(x)-f(d)}{g(x)-g(d)} - L\Big| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

which concludes the theorem. □

Remark 5.43. 1. L'Hôspital Rule can also be applied to the case when $\lim_{x\to b^-}$ replaces $\lim_{x\to a^+}$ in the theorem. Moreover, the one-sided limit can also be replaced by full limit $\lim_{x\to c}$ if $c \in (a,b)$ (by considering L'Hôspital's Rule on (a,c) and (c,b), respectively). See Example 5.44 for more details on the full limit case.

2. L'Hôspital Rule can also be applied to limits as $x\to\infty$ or $x\to-\infty$ (and here a or b has to be changed to $-\infty$ or ∞ as well). To see this, we note that if $F(x) = f\big(\frac{1}{x}\big)$ and $G(x) = g\big(\frac{1}{x}\big)$, then either $\lim_{x\to 0^+} F(x) = \lim_{x\to 0^+} G(x) = 0$ or $\lim_{x\to 0^+} F(x) = \lim_{x\to 0^+} G(x) = \infty$; thus L'Hôspital Rule implies that

$$\lim_{x\to\infty}\frac{f'(x)}{g'(x)} = \lim_{y\to 0^+}\frac{f'\big(\frac{1}{y}\big)}{g'\big(\frac{1}{y}\big)} = \lim_{y\to 0^+}\frac{f'\big(\frac{1}{y}\big)\frac{-1}{y^2}}{g'\big(\frac{1}{y}\big)\frac{-1}{y^2}} = \lim_{y\to 0^+}\frac{F'(y)}{G'(y)} = \lim_{y\to 0^+}\frac{F(y)}{G(y)} = \lim_{x\to\infty}\frac{f(x)}{g(x)}\,.$$

• **Indeterminate form** $\frac{0}{0}$

Example 5.44. Compute $\lim_{x\to 0}\frac{e^{2x}-1}{x}$.

Let $f(x) = e^{2x}-1$ and $g(x) = x$. Then f, g are differentiable on $(0,1)$ and $g(x) \neq 0, g'(x) \neq 0$ for all $x \in (0,1)$. Moreover,

$$\lim_{x\to 0^+}\frac{f'(x)}{g'(x)} = \lim_{x\to 0^+}\frac{2e^{2x}}{1} = 2$$

and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} g(x) = 0$. Therefore, L'Hôspital's Rule implies that

$$\lim_{x\to 0^+}\frac{f(x)}{g(x)} = \lim_{x\to 0^+}\frac{f'(x)}{g'(x)} = 2\,.$$

Similarly, by the fact that

1. f, g are differentiable on $(-1, 0)$ and $g(x) \neq 0, g'(x) \neq 0$ for all $x \in (-1, 0)$,
2. $\displaystyle\lim_{x \to 0^-} \frac{f'(x)}{g'(x)} = \lim_{x \to 0^-} \frac{2e^{2x}}{1} = 2$,
3. $\displaystyle\lim_{x \to 0^+} f(x) = \lim_{x \to 0^+} g(x) = 0$,

L'Hôspital's Rule implies that $\displaystyle\lim_{x \to 0^-} \frac{f(x)}{g(x)} = \lim_{x \to 0^-} \frac{f'(x)}{g'(x)} = 2$. Theorem 1.26 then shows that $\displaystyle\lim_{x \to 0} \frac{f(x)}{g(x)} = 2$ exists.