微積分 MA1001-A 上課筆記（精簡版）

2018.12.04.

Ching-hsiao Arthur Cheng 鄭經斅

Theorem 5.1: Inverse Function Differentiation

Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))} \qquad \text{for all } x \text{ with } f'(g(x)) \neq 0.$$

Definition 5.8

The function $\ln : (0, \infty) \to \mathbb{R}$ is defined by

$$\ln x = \int_1^x \frac{1}{t}\, dt \qquad \forall\, x > 0\,.$$

• $\ln : (0, \infty) \to \mathbb{R}$. is one-to-one and onto. Therefore, there exists a unique $e \in (1, \infty)$ such that $\ln e = 1$. In fact, $2 < e < 3$.

• **Logarithmic Laws**

Theorem 5.14: Logarithmic properties of $y = \ln x$

Let a, b be positive numbers and r is rational. Then

1. $\ln 1 = 0$;
2. $\ln(ab) = \ln a + \ln b$;
3. $\ln(a^r) = r \ln a$;
4. $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$.

Definition 5.25

The natural exponential function $\exp : \mathbb{R} \to (0, \infty)$ is a function defined by

$$\exp(x) = y \qquad \text{if and only if} \qquad x = \ln y\,.$$

By the definition of the natural exponential function, we have

$$\exp(\ln x) = x \quad \forall\, x \in (0, \infty) \qquad \text{and} \qquad \ln(\exp(x)) = x \quad \forall\, x \in \mathbb{R}\,. \tag{5.4.1}$$

Definition 5.26

Let $a > 0$ be a real number. For each $x \in \mathbb{R}$, the exponential function to the base a, denote by $y = a^x$, is defined by $a^x \equiv \exp(x \ln a)$. In other words,

$$a^x = \exp(x \ln a) \qquad \forall\, x \in \mathbb{R}\,.$$

Remark 5.28. The function $y = e^x$ is identical to the function $y = \exp(x)$ since

$$e^x = \exp(x \ln e) = \exp(x) \qquad \forall\, x \in \mathbb{R}\,.$$

Remark 5.29. By the definition of the natural exponential function,

$$\ln(a^x) = \ln(\exp(x \ln a)) = x \ln a \qquad \forall\, a > 0 \text{ and } x \in \mathbb{R}\,. \tag{5.4.2}$$

5.4.1 Properties of Exponential Functions

- **The law of exponentials**

(a) If $a > 0$, then $a^{x+y} = a^x a^y$ for all $x, y \in \mathbb{R}$.

(b) If $a > 0$, then $a^{x-y} = \dfrac{a^x}{a^y}$ for all $x, y \in \mathbb{R}$: Using (a), we obtain that

$$a^{x-y} a^y = a^{x-y+y} = a^x \qquad \forall\, x, y \in \mathbb{R}\,;$$

thus $a^{x-y} = \dfrac{a^x}{a^y}$ for all $x, y \in \mathbb{R}$.

(c) If $a, b > 0$, then $(ab)^x = a^x b^x$ for all $x \in \mathbb{R}$: By the definition of the exponential functions,

$$(ab)^x = e^{x \ln(ab)} = e^{x(\ln a + \ln b)} = e^{x \ln a + x \ln b} = e^{x \ln a} e^{x \ln b} = a^x b^x\,.$$

(d) If $a, b > 0$, then $\left(\dfrac{a}{b}\right)^x = \dfrac{a^x}{b^x}$ for all $x \in \mathbb{R}$: Using (b), we obtain that

$$\left(\frac{a}{b}\right)^x = e^{x \ln \frac{a}{b}} = e^{x(\ln a - \ln b)} = \frac{e^{x \ln a}}{e^{x \ln b}} = \frac{a^x}{b^x}\,.$$

(e) If $a > 0$, then $(a^x)^y = a^{xy}$ for all $x, y \in \mathbb{R}$: Using (5.4.2),

$$(a^x)^y = e^{y \ln a^x} = e^{xy \ln a} = a^{xy}\,.$$

- **The range and the strict monotonicity of the exponential functions**

Note that Theorem 5.6 implies that $\exp : \mathbb{R} \to (0, \infty)$ is strictly increasing. Suppose that $a > 1$. Then $\ln a > 0$ which further implies that

$$a^{x_1} = \exp(x_1 \ln a) < \exp(x_2 \ln a) = a^{x_2} \qquad \forall\, x_1 < x_2\,.$$

Similarly, if $0 < a < 1$, the exponential function to the base a is a strictly decreasing function.

Moreover, since $\exp : \mathbb{R} \to (0, \infty)$ is onto, we must have that for $0 < a \neq 1$, the range of the exponential function to the base a is also $\mathbb{R}$. Therefore, for $0 < a \neq 1$, the exponential function $a^{\cdot} : \mathbb{R} \to (0, \infty)$ is one-to-one and onto.

- **The differentiation of the exponential functions**

Theorem 5.30

$\frac{d}{dx}e^x = e^x$ for all $x \in \mathbb{R}$.

Proof. Define $f : (0, \infty) \to \mathbb{R}$ and $g : \mathbb{R} \to (0, \infty)$ by $f(x) = \ln x$ and $g(x) = \exp(x) = e^x$. Then f and g are inverse functions to each other, and the Inverse Function Differentiation implies that

$$g'(x) = \frac{1}{f'(g(x))} \qquad \forall\, x \in \mathbb{R} \text{ with } f'(g(x)) \neq 0\,.$$

Since $f'(x) = \frac{1}{x}$, $f'(g(x)) = \frac{1}{g(x)} = \exp(-x) \neq 0$ for all $x \in \mathbb{R}$; thus

$$g'(x) = g(x) \qquad \forall\, x \in \mathbb{R}\,.$$

□

Corollary 5.31

1. $\int_0^a e^x\,dx = e^a - 1$ for all $a \in \mathbb{R}$; 2. $\int e^x\,dx = e^x + C$.

The following corollary is a direct consequence of Theorem 5.30 and the chain rule.

Corollary 5.32

Let f be a differentiable function defined on an interval I. Then

$$\frac{d}{dx}e^{f(x)} = e^x f'(x) \qquad \forall\, x \in I\,.$$

Corollary 5.33

1. For $a > 0$, $\frac{d}{dx}a^x = a^x \ln a$ for all $x \in \mathbb{R}$ $\Big($so $\int a^x\,dx = \frac{a^x}{\ln a} + C\Big)$.

2. Let r be a real number. Then $\frac{d}{dx}x^r = rx^{r-1}$ for all $x > 0$.

3. Let f, g be differentiable functions defined on an interval I. Then

$$\frac{d}{dx}|f(x)|^{g(x)} = |f(x)|^{g(x)}\Big[g'(x)\ln|f(x)| + \frac{f'(x)}{f(x)}g(x)\Big] \qquad \forall\, x \in I \text{ with } f(x) \neq 0\,.$$

Proof. The corollary holds because $a^x = e^{x\ln a}$, $x^r = e^{r\ln x}$, and $|f(x)|^{g(x)} = e^{g(x)\ln|f(x)|}$. □

Example 5.34. $\dfrac{d}{dx}e^{-\frac{3}{x}} = e^{-\frac{3}{x}}\dfrac{d}{dx}\Big(-\dfrac{3}{x}\Big) = \dfrac{3e^{-3/x}}{x^2}$ for all $x \neq 0$.

Example 5.35. Let $f : (0,\infty) \to \mathbb{R}$ be defined by $f(x) = x^x$. Then

$$f'(x) = \frac{d}{dx}e^{x\ln x} = e^{x\ln x}\frac{d}{dx}(x\ln x) = x^x(\ln x + 1)\,.$$

Example 5.36. Find the indefinite integral $\displaystyle\int 5xe^{-x^2}\,dx$.

Let $u = -x^2$. Then $du = -2xdx$; thus

$$\int 5xe^{-x^2}\,dx = -\frac{5}{2}\int e^{-x^2}(-2x)\,dx = -\frac{5}{2}\int e^u\,du = -\frac{5}{2}e^u + C = -\frac{5}{2}e^{-x^2} + C\,.$$

Example 5.37. Compute the definite integral $\displaystyle\int_{-1}^{0} e^x\cos(e^x)\,dx$.

Let $u = e^x$. Then $du = e^x\,dx$; thus

$$\int_{-1}^{0} e^x\cos(e^x)\,dx = \int_{e^{-1}}^{1}\cos u\,du = \sin u\Big|_{u=e^{-1}}^{u=1} = \sin 1 - \sin(e^{-1})\,.$$

5.4.2 The number e

By the mean value theorem for integrals, for each $x > 0$ there exists $d \in [1, 1+x]$ such that

$$\frac{\ln(1+x)}{x} = \frac{1}{x}\int_1^{1+x}\frac{1}{t}\,dt = \frac{1}{d}$$

which implies that

$$(1+x)^{\frac{1}{x}} = \exp\big(\ln(1+x)^{\frac{1}{x}}\big) = \exp\Big(\frac{\ln(1+x)}{x}\Big) = \exp\big(\frac{1}{d}\big)\,.$$

By the fact that the natural exponential function is continuous, we find that

$$\lim_{x\to 0^+}(1+x)^{\frac{1}{x}} = \lim_{x\to 0^+}\exp\big(\frac{1}{d}\big) = \lim_{d\to 1}\exp\big(\frac{1}{d}\big) = e\,.$$

Note that the limit above also shows that $e = \displaystyle\lim_{x\to\infty}\big(1 + \frac{1}{x}\big)^x$.

5.5 Logarithmic Functions to Bases Other than e

Definition 5.38

Let $0 < a \neq 1$ be a real number. The logarithmic function to the base a, denoted by $\log_a$, is the inverse function of the exponential function to the base a. In other words,

$$y = \log_a x \qquad \text{if and only if} \qquad a^y = x\,.$$

Theorem 5.39

Let $0 < a \neq 1$. Then $\log_a x = \dfrac{\ln x}{\ln a}$ for all $x > 0$.

Proof. Let $y = \log_a x$. Then $a^y = x$; thus (5.4.2) implies that

$$y \ln a = \ln(a^y) = \ln x$$

which shows $y = \dfrac{\ln x}{\ln a}$. □

5.5.1 Properties of logarithmic functions

- **Logarithmic laws**

The following theorem is a direct consequence of Theorem 5.14 and 5.39.

Theorem 5.40: Logarithmic properties of $y = \log_a x$

Let a, b, c be positive numbers, $a \neq 1$, and r is rational. Then

1. $\log_a 1 = 0$;
2. $\log_a(bc) = \log_a b + \log_b c$;
3. $\log_a(a^x) = x$ for all $x \in \mathbb{R}$;
4. $a^{\log_a x} = x$ for all $x > 0$;
5. $\log_a \left(\frac{c}{b}\right) = \log_a c - \log_a b$.

- **The change of base formula**

We have the following identity

$$\log_a c = \frac{\log_b c}{\log_b a} \qquad \forall\, a, b, c > 0, a, b \neq 1\,.$$

In fact, if $d = \log_a c$, then $c = a^d$; thus $\log_b c = d \log_b a$ which implies the identity above.