微積分 MA1001-A 上課筆記（精簡版）

2018.11.20.

Ching-hsiao Arthur Cheng 鄭經斅

Theorem 4.31

Let $f : [a, b] \to \mathbb{R}$ be a Riemann integrable function and F be an anti-derivative of f on $[a, b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a)\,.$$

Moreover, if in addition f is continuous on $[a, b]$, then $G(x) = \displaystyle\int_a^x f(t)\,dt$ is differentiable on $[a, b]$ and

$$G'(x) = f(x) \qquad \text{for all } x \in [a, b]\,.$$

Theorem 4.33

If the function $u = g(x)$ has a continuous derivative on the closed interval $[a, b]$, and f is continuous on the range of g, then

$$\int_a^b f\big(g(x)\big)g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du\,.$$

Example 4.38. Consider the integral $\displaystyle\int_1^4 \sqrt{u}\,du = \frac{2}{3}u^{\frac{3}{2}}\Big|_{u=1}^{u=4} = \frac{2}{3}(8-1) = \frac{14}{3}$.

Let $u = g(x) = x^2$ and $a = -1$, $b = 2$ in Theorem 4.33. Then

$$\begin{aligned}\int_1^4 \sqrt{u}\,du = \int_{-1}^2 \sqrt{x^2}\cdot 2x\,dx = \int_{-1}^2 2x|x|\,dx = -\int_{-1}^0 2x^2\,dx + \int_0^2 2x^2\,dx \\ = -\frac{2}{3}x^3\Big|_{x=-1}^{x=0} + \frac{2}{3}x^3\Big|_{x=0}^{x=2} = -\frac{2}{3}(0-(-1)) + \frac{2}{3}(8-0) = \frac{2}{3}(8-1) = \frac{14}{3}\,.\end{aligned}$$

Theorem 4.34

Let g be a function with range I and f be a continuous function on I. If g is differentiable on its domain and F is an anti-derivative of f on I, then

$$\int f\big(g(x)\big)g'(x)\,dx = F\big(g(x)\big) + C$$

Letting $u = g(x)$ gives $du = g'(x)\,dx$ and

$$\int f(u)\,du = F(u) + C\,.$$

Definition 5.1

A function g is the inverse function of the function f if

$$f(g(x)) = x \quad \text{for all } x \text{ in the domain of } g \tag{5.1.1}$$

and

$$g(f(x)) = x \quad \text{for all } x \text{ in the domain of } f. \tag{5.1.2}$$

The inverse function of f is usually denoted by f^{-1}.

Some important observations about inverse functions:

1. If g is the inverse function of f, then f is the inverse function of g.

2. Note that (5.1.1) implies that

   (a) the domain of g is contained in the range of f,

   (b) the domain of f contains the range of g,

   (c) g is one-to-one: if $g(x_1) = g(x_2)$, then $x_1 = f(g(x_1)) = f(g(x_2)) = x_2$

   and (5.1.2) implies that

   (a) the domain of f is contained in the range of g,

   (b) the domain of g contains the range of f,

   (c) f is one-to-one.

   According to the statements above, the domain of f^{-1} is the range of f, and the range of f^{-1} is the domain of f.

3. A function need not have an inverse function, but when it does, the inverse function is unique: Suppose that g and h are inverse function of f, then

   (a) the domain of g is identical to the domain of h (since they are both the range of f);

   (b) for each x in the range of f,

   $$f(g(x)) = x = f(h(x))$$

   thus by the fact that f is one-to-one, $g(x) = h(x)$ for all x in the range of f.

Therefore, g and h are identical functions.

Example 5.2. The functions

$$f(x) = 2x^3 - 1 \qquad \text{and} \qquad g(x) = \sqrt[3]{\frac{x+1}{2}}$$

are inverse functions of each other since

$$f(g(x)) = 2\left[\sqrt[3]{\frac{x+1}{2}}\right]^3 - 1 = 2\frac{x+1}{2} - 1 = x$$

and

$$g(f(x)) = \sqrt[3]{\frac{2x^3-1+1}{2}} = \sqrt[3]{x^3} = x\,.$$

Theorem 5.3

A function f has an inverse function if and only if f is one-to-one.

Proof. It suffices to show the "$\Leftarrow$" direction. Suppose that f is one-to-one. Then for each x in the range of f, there exists only a unique y in the domain of f such that $f(y) = x$. Denote the map $x \mapsto y$ by g; that is,

$$y = g(x) \quad \text{if} \quad f(y) = x \text{ and } x \in \text{Range}(f)\,.$$

Then $f(g(x)) = x$ for all x in the range of f. Since the domain of g is the range of f, we find that

$$f(g(x)) = x \quad \text{for all } x \text{ in the domain of } g.$$

On the other hand, by the definition of g we must also have

$$g(f(x)) = x \qquad \text{for all } x \text{ in the domain of } f;$$

thus f has an inverse function. □

Theorem 5.4

Let f be a function with inverse f^{-1}. The graph of f contains the point (a, b) if and only if the graph of f^{-1} contains the point (b, a).

Proof. Let (a, b) be on the graph of f. Then $b = f(a)$ which implies that $f^{-1}(b) = f^{-1}(f(a)) = a$. Therefore, (b, a) is on the graph of f^{-1}. □

Remark 5.5. Theorem 5.4 implies that the graph of f and the graph of f^{-1} is symmetric above the straight line $y = x$.

Theorem 5.6

Let f be a function defined on an interval I and have an inverse function. Then

1. if f is continuous on I, then f^{-1} is continuous on its domain;
2. if f is strictly increasing on I, then f^{-1} is strictly increasing on the range of f;
3. if f is strictly decreasing on I, then f^{-1} is strictly decreasing on the range of f;
4. if f is differentiable on an interval containing c and $f'(c) \neq 0$, then f^{-1} is differentiable at $f(c)$.

Proof. We only show 2 (and the proof of 3 is similar).

To show that f^{-1} is strictly increasing on the range of f, we need to show that

$$f^{-1}(x_1) < f^{-1}(x_2) \quad \text{if } x_1 < x_2 \text{ are in the range of } f.$$

Nevertheless, if f is increasing on I and $x_1 < x_2$ are in the range of f, there exists $y_1 = f^{-1}(x_1)$ and $y_2 = f^{-1}(x_2)$ in I such that $f(y_1) = x_1$ and $f(y_2) = x_2$. Since $x_1 < x_2$, $y_1 \not\geq y_2$; thus the trichotomy law implies that $y_1 < y_2$. □

Theorem 5.7: Inverse Function Differentiation

Let f be a function that is differentiable on an interval I. If f has an inverse function g, then g is differentiable at any x for which $f'(g(x)) \neq 0$. Moreover,

$$g'(x) = \frac{1}{f'(g(x))} \qquad \text{for all } x \text{ with } f'(g(x)) \neq 0.$$

Proof. Suppose that f is differentiable at $g(c) \in I$ and $f'(g(c)) \neq 0$. We show that g is differentiable at c. If $k \neq 0$ is small enough, $g(c+k) - g(c) = h$. Then $c + k = f(g(c) + h)$. Moreover, $h \to 0$ as $k \to 0$ since g is continuous (by Theorem 5.6). Therefore,

$$\frac{g(c+k) - g(c)}{k} = \frac{h}{f(g(c)+h) - f(g(c))} = \frac{h}{f(g(c)+h) - f(g(c))}$$

which approaches $\dfrac{1}{f'(g(c))}$ as k approaches zero. Therefore, $g'(c) = \dfrac{1}{f'(g(c))}$. □

5.2 The Function $y = \ln x$（課本 §5.1）

Recall Example 4.11 that $\displaystyle\int_a^b x^q\,dx = \frac{b^{q+1} - a^{q+1}}{q+1}$ if $q \neq -1$ is a rational number and $0 < a < b$. What happened to the case $\displaystyle\int_a^b x^{-1}\,dx$? In the following, we define a new function which can be used to compute this integral.

Definition 5.8

The function $\ln : (0, \infty) \to \mathbb{R}$ is defined by

$$\ln x = \int_1^x \frac{1}{t}\,dt \qquad \forall\, x > 0\,.$$

We emphasize again that we **cannot** write $\ln x = \displaystyle\int_1^x \frac{1}{x}\,dx$ since the upper limit in the integral is some arbitrary but fixed number (denoted by x) and the variable of the integrand should be really arbitrary.

Remark 5.9. For historical reason, when the variable is clear we should ignore the parentheses and write $\ln x$ instead of $\ln(x)$. On the other hand, if the variable is product of several variables such as xy, for the sake of clarity we should still write $\ln(xy)$ instead of $\ln xy$.

5.2.1 Properties of $y = \ln x$

- **Differentiability**

Since the function $y = \dfrac{1}{x}$ is continuous on $(0, \infty)$, the Fundamental Theorem of Calculus implies the following

Theorem 5.10

$\dfrac{d}{dx} \ln x = \dfrac{1}{x}$ for all $x > 0$.

Therefore, the function $y = \ln x$ is continuous on $(0, \infty)$.

Corollary 5.11

The function $\ln : (0, \infty) \to \mathbb{R}$ is strictly increasing on $(0, \infty)$, and the graph of $y = \ln x$ is concave downward on $(0, \infty)$.

Example 5.12. In this example we prove that

$$x - \frac{x^2}{2} \leqslant \ln(1 + x) \leqslant x \qquad \forall\, x > 0\,. \tag{5.2.1}$$

Let $f(x) = \ln(1 + x) - x + \dfrac{x^2}{2}$ and $g(x) = \ln(1 + x) - x$. Then for $x > 0$,

$$\begin{aligned} f'(x) &= \frac{1}{1 + x} - 1 + x = \frac{x^2}{1 + x} > 0\,, \\ g'(x) &= \frac{1}{1 + x} - 1 = \frac{-x}{1 + x} < 0\,. \end{aligned}$$

The two identities above shows that f is strictly increasing on $[0, \infty)$ and g is strictly decreasing on $[0, \infty)$. Therefore,

$$f(x) > f(0) = 0 \quad \text{and} \quad g(x) < g(0) = 0 \qquad \forall\, x > 0\,.$$

These inequalities lead to (5.2.1).